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Perturbative analysis of the Conformal Constraint Equations in General Relativity

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Submitted in partial fulfilment of the requirements for the degree of
Doctor of Philosophy

Declaration

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The content of this thesis is based partly on the following papers:

- Valiente Kroon, J.A. and Williams, J.L., "Dain s invariant on non-time symmetric initial data sets", *Classical and Quantum Gravity*, 34.12 (2017): 125013.
- Valiente Kroon, J.A. and Williams, J.L., "A perturbative approach to the construction of initial data on compact manifolds", *arXiv preprint*, arXiv:1801.07289 (2018), (Submitted to *Communications in Mathematical Physics*).

Abstract

The main purpose of this thesis is to develop a perturbative method for the construction of initial data for the Cauchy problem in General Relativity. More precisely, it considers the problem of constructing solutions to the so-called Extended Constraint Equations (ECEs), based on the method of A. Butscher and H. Friedrich. For much of the thesis, attention is restricted to closed initial hypersurfaces —that is to say, initial data for cosmological spacetimes. In doing so, it is possible to study the potential obstructions to the implementation of the "Friedrich–Butscher method" in a more systematic manner. The central result of this thesis is that initial data describing certain spatially-closed analogues of the Friedmann–Lemaître–Robinson–Walker (FLRW) spacetime are suitable background initial data sets on which to apply the Friedrich Butscher method. That is to say, one can construct solutions of the ECEs as non-linear perturbations of these background geometries, for which certain components of the extrinsic curvature and of the Weyl curvature (of the resulting spacetime development) are prescribed at the outset. Progress is then made towards identifying a broader class of admissible background geometry, and a streamlined version of the method is proposed which overcomes some of the difficulties inherent to the earlier approach. An elliptic reduction of the full Conformal Constraint Equations of H. Friedrich is then described, and the earlier analysis of the spatially-closed FLRW background geometries is generalised in this context. The last part of the thesis concerns the separate (though not altogether unrelated) problem of Killing Initial Data sets, and how they may be generalised to describe a notion of approximate spacetime Killing symmetry, at the level of the initial data. This builds on work of S. Dain, which is extended from the time symmetric case to the generic asymptotically-Euclidean case.

To my parents:

Andrea and Gerwyn

Acknowledgements

Foremost, I would like to thank my PhD supervisor Dr Juan A. Valiente Kroon for his guidance and support throughout my PhD. His inexhaustible patience and generosity with his time and ideas have been invaluable, and greatly appreciated. I have also benefited from interactions with the other members of the GR group at QMUL, which I have had the privilege of seeing expand over the last few years. In particular, it has a been a pleasure to get to know the other students of the group: Michael Cole, Edgar Gasperín, Adem Hursit, Diego Carranza, Mikael Normann, Peng Zhao and Liam Williams, from whom I have learnt a lot, and who have made my time at QMUL much more enjoyable. I would especially like to thank Christopher Ablitt, Liam Williams, Zoe Wyatt and Andrea Devereaux for reading various parts of this thesis and for their suggestions for its improvement. I am also very grateful to Dr Shabnam Beheshti for her mentorship; her advice, both professional and personal, has been indispensable over the course of my PhD. I would also like to take this opportunity to thank my examiners Prof. Luc Nguyen and Dr Huy Nguyen in advance for reading this thesis.

Special thanks go to my family, in particular my parents, for their constant support and belief in me, and without whom I would not be where I am today. Last but not least, I owe a huge debt of gratitude to Christopher Ablitt for having always been there for me, and for encouraging me when I needed it most.

Contents

Chapter 1

Introduction

1.1 General Relativity

The advent of Einstein's theory of General Relativity (GR) marked a paradigm shift in our approach towards understanding the fundamental forces of nature. In GR, Newton's notion of a gravitational force is abandoned in favour of a more geometric viewpoint, in which gravitation is described as a manifestation of the *curvature* of a 4-dimensional object called *spacetime*[1]. In mathematical terms, the spacetime consists of a Lorentzian manifold $(\mathcal{M}, \boldsymbol{g})$ —that is to say, a 4-dimensional differentiable manifold $\mathcal{M}$ and a metric tensor $\boldsymbol{g}$ of signature 2— satisfying the Einstein field equations:

$$R_{ab} - \tfrac{1}{2} R g_{ab} + \lambda g_{ab} = T_{ab}.$$

See [1], for example. Here, R_{ab} denotes the *Ricci curvature tensor*, $R \equiv g^{ab} R_{ab}$ the *Ricci scalar curvature*, λ the *cosmological constant* and T_{ab} the *energy-momentum tensor*. The content of these equations is captured succinctly in the often-quoted statement of John Archibald Wheeler:

> "Spacetime tells matter how to move; matter tells spacetime how to curve."

Rather than 3-dimensional space acting simply as a stage on which the laws of physics play out over time, the two are united in a single spacetime, which is itself the fundamental object of study. General Relativity is a highly successful theory of gravity, having passed all experimental tests so far. Indeed, in 2016 the LIGO collaboration announced the first direct detection of gravitational waves, originating from the inward spiral and merger of two stellar-mass black holes, [2]. This constitutes the first experimental verification of GR in the strong regime.

One of the key properties of the Einstein field equations, and indeed the source of much of the difficulty in their mathematical analysis, is that they are highly non-linear. In an arbitrary coordinate system, x^μ, the Ricci curvature can be written as follows

$$R_{\mu\nu} = -\tfrac{1}{2} g^{\rho\lambda} \frac{\partial^2 g_{\mu\nu}}{\partial x^\mu \partial x^\nu} + g_{\rho(\mu} \nabla_{\nu)} \Gamma^\rho + H(\boldsymbol{g}, \partial \boldsymbol{g})_{\mu\nu}$$

where $\Gamma^\mu \equiv g^{\nu\rho} \Gamma^\mu_{\nu\rho}$ and $H(\boldsymbol{g}, \partial\boldsymbol{g})_{\mu\nu}$ are functions of the metric coefficients and their first derivatives. In physical terms, this reflects the fact that the gravitational field does not obey a superposition principle —the gravitational "force" exerted by two objects on a third is not simply the sum of two individual contributions. This is the property which allows for new phenomena such as black holes,

[1] In this thesis, we will only be concerned with GR in 4 dimensions.

not predicted in Newtonian gravity, and which are possible even in the absence of matter —the gravitational field is “self-sourcing”. In this thesis we will restrict to the vacuum case, in which the Einstein field equations reduce to

$$R_{ab} = \lambda g_{ab}.$$

Another one of the defining features of GR is that it is *generally covariant* —that is to say, the field equations are invariant under coordinate transformations. This reflects the postulate that physical laws should be independent of the frame of reference (or observer) with respect to which they are measured. This also leads to mathematical difficulties, since some of the key properties of the Einstein field equations, as a system of Partial Differential Equations, only become apparent upon fixing a suitable “gauge” (i.e. coordinate system). The above issues are especially pertinent within in framework of the *Cauchy problem* in GR, to which we now turn, in which the hyperbolicity of the Einstein field equations (in a suitable gauge) is central to obtaining an well-posed initial value problem.

1.2 The Cauchy problem in GR

The Cauchy problem for the Einstein field equations is a cornerstone of Mathematical Relativity. Indeed, the proper rigorous formulation of many of the outstanding problems in mathematical Relativity, such as the *stability* of certain special solutions, are made within this framework. In the Cauchy problem, one considers a foliation of the spacetime manifold by codimension-1 spacelike hypersurfaces, with respect to which the Einstein field equations decompose into two subsystems —see e.g. [1, 3]. The first system, the *Einstein constraint equations* (consisting of the *Hamiltonian* and *momentum* constraints) which are intrinsic to the leaves of the foliation. The second system, the *evolution equations*, govern how geometric information is propagated from one leaf to the next. The Hamiltonian and Momentum constraints in vacuum are given, respectively, by

$$r[\boldsymbol{h}] + K^2 - K_{ij}K^{ij} - 2\lambda = 0, \tag{1.2.1a}$$

$$D^iK_{ij} - D_jK = 0, \tag{1.2.1b}$$

where $r[\boldsymbol{h}]$ denotes the Ricci scalar curvature of the metric $\boldsymbol{h}$, and $K \equiv h^{ij}K_{ij}$. A solution —an *initial data set*— consists of 3-dimensional manifold $\mathcal{S}$, a Riemannian metric h_{ij}, and a symmetric two-tensor K_{ij}. The latter, called the *extrinsic curvature*, describes the geometry of the embedding $\mathcal{S} \hookrightarrow \mathcal{M}$.

There are several approaches to the construction of evolution equations. One approach is to reduce the Einstein field equations to a second-order hyperbolic system through the use of *wave* (or *harmonic*) *coordinates* —i.e. coordinates x^μ satisfying

$$\Box x^\nu \equiv \nabla_\mu \nabla^\mu x^\nu = 0.$$

A short computation then shows that $\Gamma^\mu = \Box x^\mu = 0$, and hence in such a coordinate system the vacuum Einstein field equations reduce to

$$-\tfrac{1}{2} g^{\rho\lambda} \frac{\partial^2 g_{\mu\nu}}{\partial x^\mu \partial x^\nu} + H(\boldsymbol{g}, \partial \boldsymbol{g})_{\mu\nu} - \lambda g_{\mu\nu} = 0, \tag{1.2.2}$$

which is manifestly hyperbolic. One can show using the Bianchi identity that

$$\Box\Gamma^\mu + R_{\mu\nu}\Gamma^\nu = 0.$$

Hence, it follows that if

$$\Gamma^\mu\big|_{\mathcal{S}} = 0, \qquad \boldsymbol{n}(\Gamma^\mu)\big|_{\mathcal{S}} = 0,$$

where $\boldsymbol{n}$ denotes the normal to the foliation, then the gauge is "propagated" —i.e. $\Gamma^\mu = 0$ in the resulting *spacetime development* (see below), by virtue of the uniqueness of solutions to hyperbolic initial value problems. The condition $\boldsymbol{n}(\Gamma^\mu)\big|_{\mathcal{S}} = 0$ is essentially guaranteed by the constraint equations. Hence, if one constructs a solution $\boldsymbol{g}$ to equation (1.2.2) with initial data $(\boldsymbol{h}, \boldsymbol{K})$ solving the constraint equations on some *initial hypersurface* $\mathcal{S}$, in a gauge satisfying $\Gamma^\mu|_{\mathcal{S}} = 0$, then $\boldsymbol{g}$ indeed satisfies the Einstein field equations. This is captured in the well-known local-existence result of Choquet-Bruhat [4]:

Theorem. Given an initial data set $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$ there exists a vacuum spacetime $(\mathcal{M}, \boldsymbol{g})$, the *local spacetime development*, such that $\mathcal{S}$ is a spacelike hypersurface of $\mathcal{M}$ and $\boldsymbol{h}, \boldsymbol{K}$ are the intrinsic metric and extrinsic curvature (i.e. first and second fundamental forms) induced by $\boldsymbol{g}$ on $\mathcal{S}$.

Moreover, it can be shown [5] that there exists a unique *maximal* development $(\mathcal{M}, \boldsymbol{g})$, in a precise sense, called the *maximal globally hyperbolic development*.

Rather than working with the above second-order equations, the evolution equations may be re-expressed as a first-order system in the form of the ADM (Arnowitt–Deser–Misner) equations, [6]:

$$\partial_t \begin{pmatrix} \gamma_{ij} \\ K_{ij} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} D\Phi^* \begin{pmatrix} N \\ X_i \end{pmatrix}$$

for a given *lapse-shift pair*, N, X^i, in terms of which $\partial_t = N\boldsymbol{n} + \boldsymbol{X}$. See [7] for an introduction. Here, $D\Phi^*$ denotes the formal adjoint of the linearisation of the *constraint map*

$$\Phi \begin{pmatrix} h_{ij} \\ K_{ij} \end{pmatrix} \equiv \begin{pmatrix} r[\boldsymbol{h}] + K^2 - K_{ij}K^{ij} - 2\lambda \\ -D^i K_{ij} + D_j K \end{pmatrix}.$$

One of the interesting features of the ADM equations is that they realise the Einstein field equations as a flow generated by the constraint equations.

In addition to the being able to construct initial data sets, one would like to know how various properties of the resulting spacetime developments are encoded at the level of their initial data. Properties of interest include their global/asymptotic properties, the formation of singularities and the existence of *Killing symmetries*. It turns out that the latter is intimately connected with the ADM formalism. More precisely, vanishing of the right-hand-side of the ADM equations equivalent to the so-called *Killing Initial Data* (or *KID*) equations. The existence of a solution to the KID equations is a necessary and sufficient condition for there to exist a Killing vector on the spacetime development of $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$. More precisely, a solution (N, X^i) describes the normal and parallel components (on $\mathcal{S}$) of a spacetime Killing vector —see [8–10]. We will come back to the KID equations in Chapters 6.

As mentioned previously, one of the central outstanding problems in the field of mathematical Relativity is that of stability. In particular, under which conditions can we guarantee that a gravitating system will settle down to a stationary (equilibrium) regime? In order to be able to address such questions, one first needs to be able to identify the spacetime which is anticipated to lie at the endpoint of the evolution, and to have a way of quantifying deviation from that spacetime. One approach to this problem was given by S. Dain in [11], in which the author defines a notion of *approximate Killing vectors*, arising as the solution of a certain fourth-order elliptic equation. This approach has also been adopted in the context of *Killing spinors* —see [12, 13]. We will return to this problem in Chapter 8.

1.3 Methods of constructing initial data

Given the advent of gravitational wave astronomy it is now important, perhaps more than ever, to be able to construct physically-relevant initial data sets. In practice, however, the problem of constructing initial data is a difficult one, owing to the non-linearity and highly-coupled nature of the constraint equations. Moreover, unlike the Einstein field equations which comprise ten independent equations for the ten components of the 4-metric, g_{ab}, the constraint equations are highly underdetermined. Indeed, they consist of only four equations for twelve unknowns —the components of the 3-metric, h_{ij}, and of the extrinsic curvature, K_{ij}. Consequently, any attempt to construct solutions of the constraint equations first requires a choice of *freely-prescribed* and *determined* fields. This results in a substantial variation in the methods of initial data construction, each of which offers a different viewpoint on the classical problem of describing the *degrees of gravitational freedom* —i.e. how should one parametrise the space of solutions to the Einstein field equations, if indeed such a parametrisation exists? Note that by the theorem of Choquet-Bruhat, the problem of describing the space of solutions to the Einstein field equations is reduced, in some sense, to the analogous problem for the constraint equations.

Perhaps the most developed of the methods for solving the Einstein constraints are the "Conformal Method" of Lichnerowicz–Bruhat–York (see e.g. [14]), and the "gluing" constructions of Corvino–Schoen — [15, 16]. For an overview of these methods, we refer the reader to [7]. Both techniques rely on reformulating the constraint equations as systems of elliptic PDEs. While it is often stated that the constraints are elliptic, they are *a priori* of no particular PDE character (elliptic, hyperbolic nor parabolic). Indeed, it is worth noting in passing that it is also possible to recast the constraint equations as a symmetric-hyperbolic(-parabolic) system, and they can therefore themselves be treated as an evolution problem —see [17–19].

In the Conformal Method, one prescribes the conformal class of the spatial metric, $[\boldsymbol{h}]$, along with the mean extrinsic curvature, $K \equiv h^{ij}K_{ij}$, and the TT part (see Section 2.1.3) of the *York-scaled* extrinsic curvature[2]. In doing doing so, the constraint equations are rendered elliptic for the remaining determined fields, which consist of a scalar field (the conformal factor) and a covector field. One can therefore appeal to methods of elliptic PDE theory, such as the maximum principle, Fredholm alternative and the method of sub/super solutions, in their analysis. One of the features of the conformal method is that the free data are *unphysical*, in the sense that one needs to solve the full system of (conformally formulated) constraint equations (solving in particular for the conformal factor) before one can obtain their physically-meaningful counterparts. For closed $\mathcal{S}$ and constant K (i.e. *constant mean curvature*, or *CMC*, initial data), the admissible choices of free data have been fully determined —see [7] for an account. It is interesting to note that this requires the use of the positive resolution of the Yamabe problem [20]. In the case of non-constant K, however, little is known unless one assumes K to be "near-constant" —see [21–23], for example. One immediate restriction on the free data occurs in the case that $\mathcal{S}$ is closed and $[\boldsymbol{h}]$ admits a conformal Killing

[2]There are several variations of the method. We are describing here what is sometimes referred to as "Method A"

vector, V^i: one sees that $\boldsymbol{V}(K)$ must integrate to zero over $\mathcal{S}$ as a consequence of the momentum constraint. While the existence of a conformal Killing vector field is problematic, it is not clear whether it poses an essential obstruction; see [24] for an attempts to remove the assumption of non-existence of conformal Killing fields. In [25], it is argued that the Conformal Method provides the correct way of parametrising the solution space of the constraints. However, recent works have demonstrated issues of non-uniqueness of solutions within the conformal method, for certain choices of free data —see [26, 27], for instance.

In this thesis we will consider an alternative approach based on [28, 29], which we call the *Friedrich–Butscher* method. The approach is again based on elliptic PDE theory, though is fundamentally perturbative, with solutions being constructed as non-linear perturbations of a given *background initial data set* via the Implicit Function Theorem. The method identifies yet another choice of free and determined fields; in fact, since the method makes use of an *extended* version of the constraint equations —the *Extended Constraint Equations* (or *ECEs*)— the prescribed fields include certain components of the Weyl curvature (roughly speaking, the TT components of the *electric* and *magnetic* parts). This method therefore affords the possibility of constructing initial data with certain properties of the gravitational radiation prescribed at the outset. Note that since the method is not based on a conformal reformulation of the constraints, the free data are physical in the sense of determining, *a priori*, physically relevant properties of the initial data set and its resulting spacetime development. This method therefore potentially offers a new perspective on the gravitational degrees of freedom problem.

1.4 Conformal methods in GR

Applications of conformal methods to problems in General Relativity have a long history, beginning perhaps in 1944 with Lichnerowicz's work on the "Conformal Method" [30], which was outlined in the previous section. More generally, the power of conformal methods in GR stems from the fact that conformal transformations $\tilde{\boldsymbol{g}} \to \boldsymbol{g} \equiv \Theta^2 \tilde{\boldsymbol{g}}$ allow one to study the global properties of a spacetime geometry while preserving its *causal structure* (i.e. light-cone structure). This is the basic idea behind Penrose's notion of *asymptotic simplicity*, which he introduced in [31]. Roughly speaking, a spacetime is *asymptotically simple* if it admits a conformal extension in a way that is analogous to one of the three constant-curvature spacetimes: Minkowski, de Sitter or anti-de Sitter space. For such a spacetime, one can hope to analyse the asymptotic properties of its gravitational field in a systematic way, via its conformal extension.

However, while the Weyl tensor $C^a{}_{bcd}$, which captures the gravitational radiation of a spacetime, is conformally-invariant, the Ricci curvature is less well behaved under conformal transformations. As a result, the Einstein field equations degenerate at the "conformal boundary", $\mathscr{I}$, where the conformal factor Θ vanishes. In order to deal with this problem, H. Friedrich introduced his *Conformal Field Equations*, or *CFEs* [32, 33], which can be thought of as a conformally-covariant version of the Einstein field equations which is regular all the way up to $\mathscr{I}$. The CFEs have been shown to admit a hyperbolic reduction, in much the same way as the Einstein field equations. This provides a systematic framework for the implementation of Penrose's proposal. The reader is referred to [34] for more details on this, and conformal methods in general.

Just as the Einstein field equation give rise to a system of constraint equations (the Einstein constraint equations) under a $3 + 1$ decomposition, so too do the CFEs. This system, referred to here as the *Conformal Constraint Equations*, is a much larger system of equations than the Einstein constraints and has a much more complicated algebraic structure. That being said, the CCEs are in fact entirely equivalent to the Einstein constraints, in the sense that a solution of the Einstein constraints corresponds precisely to a family of conformally-related solutions of the CCEs.

This formalism was used to give the first global non-linear stability result for the de Sitter spacetime and the first *semi-global* non-linear stability result for the Minkowski spacetime based on *hyperboloidal foliations* —see [35, 36]. In order to have a more complete picture, one would like to be able to construct hyperboloidal data directly as solutions to the CCEs. However, a full understanding of the CCEs presents a significant mathematical challenge. As mentioned previously, the ECEs can be thought of as a simplification of the CCEs, and therefore an understanding of the former is a necessary first step is to understanding the latter. While the conformal aspect of the equations is lost in restricting to the ECEs, several of the important features of the CCEs are retained. In particular, the ECEs can be thought of as being comprised of two primary equations and two integrability conditions, the interplay of which is fundamental to Butscher's approach in [28,29].

1.5 Thesis overview

Chapter 2 collects together some mathematical preliminaries. Section 2.1 recalls some basic concepts of differential geometry, including the relevant concepts of submanifold geometry, in addition to the definitions of various tensor spaces and their decompositions. Section 2.2 recalls some useful conformal transformation formulae and Section 2.3 collects together some useful results from functional analysis and elliptic PDE theory on closed manifolds.

Chapter 3 introduces the main equations of interest in the remainder of the thesis. Section 3.1 briefly describes the Conformal Field Equations (CFEs) of H. Friedrich. Section 3.2 then introduces the Conformal Constraint Equations (CCEs), with a discussion of various properties of importance, including their relation to the Einstein constraint equations, their quasi-conformal covariance and the interdependence of the equations as encoded by certain integrability conditions. Section 3.3 explores various simplifications of the CCEs, most importantly the Extended Constraint Equations (ECEs), and recalls the method of A. Butscher, [28, 29], for the construction of initial data sets as perturbations of flat initial data.

Chapter 4 presents a first application of what will be called the Friedrich–Butscher method, to the construction of initial data for cosmological spacetimes. Section 4.1 introduces the method, outlining the main ideas. These include the construction of an auxiliary elliptic system, the *sufficiency argument* and a first discussion of the geometric *obstructions* to the implementation of the method. Section 4.2 describes a class of *background initial data* on which the method may be successfully implemented; this is carried out in Section 4.3, the main result being given in Theorem 2. Such initial data may be thought of as CMC initial data for a spatially compact analogue of the "$k = -1$" *Friedmann–Lemaître–Robinson–Walker* spacetime. Finally, Section 4.4 provides an explicit construction and parametrisation of the freely prescribed data of the method, for such background initial data.

Chapter 5 is concerned with applying the Friedrich–Butscher method to more general background geometries than those considered in Chapter 4. Section 5.1 presents the Friedrich–Butscher method in more generality than considered previously, involving a slightly modified gauge-fixing procedure. Section 5.2 establishes conditions, (C1)–(C4), under which the method can be implemented, the main result of this section being Theorem 3. Section 5.3 provides a first exploration of conditions (C1)–(C4), in particular their relationship to notions of *curvature pinching* —see Corollaries 2 and 3.

Chapter 6 presents an alternative version of the Friedrich–Butscher method, making use of an inbuilt mixed-order ellipticity of the ECEs, and a generalisation of the de Turck trick for the elliptic reduction of the Ricci operator. In Section 6.1 the ellipticity of the ECEs is revisited. Section 6.2 introduces a new system of auxiliary equations, and gives a first investigation of the space of obstructions to their integrability, which now include the KID sets. Section 6.3 then applies the new method to the construction of initial data as perturbations of time symmetric ($K = 0$) initial data sets, the main result being Theorem 6 which is an improvement on Theorem 3 of the previous chapter.

Chapter 7 describes an extension of the Friedrich–Butscher method to the full system of CCEs. Section 7.1 discusses an elliptic reduction of the CCEs, involving the identification of freely prescribed and determined fields. As proof-of-concept, Section 7.2 then applies the method to the background initial data sets of Chapter 4, now considered as solutions to the full CCEs (with trivial conformal factor). The main result, given in Theorem 7, is the existence of non-linear perturbative solutions of the full CCEs. Finally, Section 7.3 presents a heuristic discussion of the CCEs as an elliptic boundary value problem.

Chapter 8 concerns the separate, though tangentially related problem of generalising the notion of Killing symmetries in the context of GR, via so-called *approximate KID sets.* This is based on the work of [11]. Section 8.1 introduces the approximate Killing Initial Data equations and the required theory of elliptic operators on weighted Sobolev spaces, before proving the existence of solutions —approximate KID sets— in Theorem 8, for generic asymptotically–Euclidean data sets. Section 8.3 then further explores the asymptotics of the approximate KID set for conformally flat initial data, and Section 8.4 expresses the resulting *Dain invariant* in terms of a bulk integral over the initial hypersurface.

Many of the computations in this thesis were facilitated through the use of the xAct suite in Mathematica —information and documentation on the xAct suite can be found on “www.xact.es”. The packages xTensor and xPerm —see [37]— were used throughout the thesis to perform abstract tensorial computations, and the package xPert —see [38]— was used to compute the linearisations of various operators in Chapters 5 and 6.

Chapter 2

Mathematical preliminaries

The purpose of this chapter is to remind the reader of some concepts of Differential/Conformal Geometry and elliptic PDEs that will be needed for the remainder of this thesis, in addition to fixing some conventions.

Notation: Where convenient we make use of index-free notation in which tensorial objects are written in boldface.

2.1 Differential geometry

We begin with the definition of a manifold:

Definition 1. Let $\mathcal{M}$ be a topological space, and let $\mathcal{U} \subseteq \mathcal{M}$ and $\mathcal{V} \subseteq \mathbb{R}^n$ be open subsets. Let $\varphi : \mathcal{U} \to \mathcal{V}$ be a homeomorphism, then the pair $(\mathcal{U}, \varphi)$ is called a *chart*. A collection of charts, $\{\mathcal{U}_\alpha, \varphi_\alpha\}$ such that $\mathcal{U}_\alpha$ form an open cover of M is called an *atlas*. A topological space $\mathcal{M}$ is an *n-dimensional manifold* if

1. $\mathcal{M}$ is paracompact and Hausdorff;
2. $\mathcal{M}$ admits an atlas $\{U_\alpha, \varphi_\alpha\}$, $\varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^n$.

Moreover, $\mathcal{M}$ is said to be a C^k-*differentiable manifold* if the transition maps,

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta),$$

are $k-$times differentiable diffeomorphisms. Finally, $\mathcal{M}$ is said to be a *smooth* manifold if the transition maps are C^∞.

Remark 1. Throughout this thesis, all manifolds will be implicitly assumed to be connected and orientable.

When equipped with a non-degenerate metric tensor $\boldsymbol{g}$, $(\mathcal{M}, \boldsymbol{g})$ is called a pseudo-Riemannian manifold. The resulting volume measure will be denoted $d\mu_{\boldsymbol{g}}$, or $d\mu$ when the metric is clear from context.

2.1.1 Derivatives, Curvature and the Bianchi identities

For the purposes of this section, let $(\mathcal{M}, \boldsymbol{g})$ denote a general pseudo-Riemannian manifold. In a given local coordinate system, x^μ, the *Christoffel symbols* are given by

$$\Gamma^\rho_{\mu\nu} = \tfrac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}).$$

The *Levi-Civita connection* acts as $\nabla_a f = \partial_a f$ on scalar functions $f : M \to \mathbb{R}$,

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\lambda\mu} V^\lambda$$

on vector fields and is extended to all other tensor fields via the Leibniz rule. The Lie derivative along a given vector field $\boldsymbol{W}$, denoted $\mathcal{L}_{\boldsymbol{V}}$, acts on scalar functions as $\mathcal{L}_{\boldsymbol{W}} f = V^\nu \partial_\nu f$, on vectors fields as

$$\mathcal{L}_{\boldsymbol{W}} V^\mu = W^\nu \partial_\nu V^\mu - (\partial_\rho W^\mu) V^\rho,$$

and is extended to all other tensor fields by again imposing the Leibniz property. The partial derivatives may be replaced with covariant derivatives —in particular,

$$\mathcal{L}_{\boldsymbol{V}} g_{ab} = \nabla_a V_b + \nabla_b V_a.$$

Our conventions for the Riemann curvature tensor, $R^a{}_{bcd}$, are fixed by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) v^c = R^c{}_{dab} v^d,$$

and the covariant form is given by $R_{abcd} = g_{ae} R^e{}_{bcd}$. The Ricci curvature tensor is given by $R_{ab} \equiv R^c{}_{acb}$ and the Ricci scalar curvature is given by $R \equiv g^{ab} R_{ab}$. As is customary, we will use round and square brackets to note symmetrisation and anti-symmetrisation, respectively

$$T_{(ab)c\cdots d} = \tfrac{1}{2}(T_{abc\cdots d} + T_{bac\cdots d}), \qquad T_{[ab]c\cdots d} = \tfrac{1}{2}(T_{abc\cdots d} - T_{bac\cdots d}).$$

The Riemann curvature enjoys the following symmetries

$$R_{abcd} = -R_{bacd}, \tag{2.1.1}$$
$$R_{abcd} = R_{cdab}, \tag{2.1.2}$$
$$R_{a[bcd]} = 0, \tag{2.1.3}$$

the last of which is called the *first Bianchi identity*. The *second Bianchi identity* is given by

$$\nabla_{[a} R_{bc]de} = 0.$$

Tracing with the metric $\boldsymbol{g}$, one obtains the *contracted (second) Bianchi identity*

$$\nabla^b R_{ab} - \tfrac{1}{2} \nabla_a R = 0.$$

The *Weyl curvature tensor*, denoted $C^a{}_{bcd}$, is the completely tracefree part of the Riemann curvature and enjoys the same symmetries as the Riemann tensor, (2.1.1)–(2.1.3). It will also prove convenient to define the *Schouten curvature tensor*, which in dimension $n \geq 2$ is given by

$$L_{ab} \equiv \frac{1}{n-2}\left(R_{ab} - \frac{R}{2(n-1)}g_{ab}\right).$$

The Schouten tensor clearly encodes the same information as the Ricci tensor, but is more convenient for use in the *Kulkarni–Nomizu* decomposition:

$$R_{abcd} = C_{abcd} + (\boldsymbol{g} \owedge \boldsymbol{L})_{abcd} \equiv C_{abcd} + 2g_{a[c}L_{d]b} - 2g_{b[c}L_{d]a}$$

Here, the operation $\owedge$ (defined by the right hand side) is called the *Kulkarni–Nomizu product.* The Schouten tensor is also convenient from the perspective of conformal geometry —see Section 2.2.

Notation: From now on, unless otherwise stated, the notation $(\mathcal{M}, \boldsymbol{g})$ will be reserved for 4-dimensional Lorentzian metrics of signature $(-,+,+,+)$, with Levi-Civita connection denoted by ∇ and the resulting curvature quantities denoted with capital kernel letters —e.g. R, R_{ab}, R_{abcd}, L_{ab}. The notation $(\mathcal{S}, \boldsymbol{h})$ will be reserved for 3-dimensional Riemannian manifolds, the Levi-Civita connection will be denoted by D and the curvature quantities denoted by r, r_{ij}, r_{ijkl}, l_{ij} —note here that we are also using the Latin characters $i, j, k, \ldots$ to denote abstract tensorial indices. When we wish to emphasise the role of the Ricci curvature as a differential operator acting on the metric, we will write $\mathrm{Ric}[\cdot]$, while for the scalar curvature we will write $r[\boldsymbol{h}]$, $R[\boldsymbol{g}]$ in dimensions 3 and 4, respectively.

In dimension 3, the Weyl curvature is trivial and so the Kulkarni–Nomizu decomposition reduces to

$$r_{ijkl} = (\boldsymbol{h} \owedge \boldsymbol{l})_{ijkl} \equiv 2h_{i[k}l_{l]j} - 2h_{j[k}l_{l]i}.$$

2.1.2 Projector formalism and submanifold geometry

We begin with the following definition:

Definition 2. Let $(\mathcal{M}, \boldsymbol{g})$ be a Lorentzian manifold. A subset $\mathcal{S} \subseteq \mathcal{M}$ is said to be a a *Cauchy surface* of $\mathcal{M}$ if no two of its points can be connected by a timelike curve *and* if any inextendible causal curve in $\mathcal{M}$ meets $\mathcal{S}$. If $(\mathcal{M}, \boldsymbol{g})$ possesses a Cauchy surface, then it is said to be *globally hyperbolic.*

It follows from the definition (see [3, 39]) that a Cauchy surface $\mathcal{S}$ is automatically a C^0 hypersurface of $\mathcal{M}$ and that $\mathcal{M}$ possesses a *global time function* —i.e. a function $t : \mathcal{M} \to \mathbb{R}$ for which $\nabla^a t$ is timelike vector. Hence, the existence of a Cauchy surface implies that $\mathcal{M}$ has the topology of $\mathbb{R} \times \mathcal{S}$ —i.e. that there is a foliation

$$\mathcal{M} = \bigcup_{t \in \mathbb{R}} \mathcal{S}_t \tag{2.1.4}$$

where each *leaf* of the foliation, $S_t \equiv \{p \in \mathcal{S} \mid t(p) = t\}$, is diffeomorphic to $\mathcal{S}$. By construction, any solution of the Cauchy problem —i.e. any *spacetime development* of a given initial data set $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$— is a globally hyperbolic manifold with Cauchy hypersurface $\mathcal{S}$. We note in passing that not all solutions of interest are globally hyperbolic —e.g. the anti-de Sitter spacetime.

The *projector* associated to the foliation is given by

$$h_a{}^b = \delta_a{}^b + n_a n^b$$

with $n_a \equiv g_{ab}n^b$ the *unit co-normal* to $\mathcal{S} \hookrightarrow \mathcal{M}$. It is straightforward to check that ${h_a}^b$ is indeed idempotent —that is to say, that ${h_a}^c{h_c}^b = {h_a}^b$. The projector can be used to perform a "$3+1$" decomposition of any given tensor field over $\mathcal{M}$. For instance, given a 2-tensor F_{ab},

$$\begin{aligned} F_{ab} = {\delta_a}^c{\delta_b}^d F_{cd} &= ({h_a}^c - n_a n^c)({h_b}^d - n_b n^d)F_{cd} \\ &= F^{\parallel}_{ab} - n_a F^{\perp\parallel}_b - n_b F^{\parallel\perp}_a + n_a n_b F^{\perp}, \end{aligned}$$

where $F^{\parallel}_{ab} \equiv {h_a}^c{h_b}^d F_{cd}$, $F^{\perp\parallel}_b \equiv n^c{h_b}^d F_{cd}$, $F^{\parallel\perp}_b \equiv n^d{h_b}^c F_{cd}$ and $F^{\perp} \equiv n^a n^b F_{ab}$, all of which are *spatial* —that is to say, $\boldsymbol{F}^{\parallel}, \boldsymbol{F}^{\perp\parallel}$ and $\boldsymbol{F}^{\parallel\perp}$ are all $\boldsymbol{n}$-orthogonal, and may therefore be identified with tensor fields over $\mathcal{S}$.

The induced metric on $\mathcal{S}$ is given by

$$h_{ab} \equiv g_{bc}{h_a}^c \equiv g_{ab} + n_a n_b,$$

while the tensor

$$h^{ab} \equiv g^{ac}{h_c}^b \equiv g^{ab} + n^a n^b$$

plays the role of the inverse of h_{ab}, when restricted to act on $T^{\bullet}\mathcal{S}$. Note that though the induced metric is given here in terms of spacetime indices, $a, b, \cdots$, it can naturally be identified with an element of $T\mathcal{S} \otimes T\mathcal{S}$ since

$$h_{ab}n^b = h_{ba}n^b = g_{ab}n^b + n_a n_b n^b = n_a - n_a = 0.$$

The Levi–Civita connection of h_{ab}, acting on a spatial covector $v_a \in T^*\mathcal{S}$, is then given in terms of ∇ as follows

$$D_a v_b = {h_a}^c{h_b}^d \nabla_c v_d.$$

The *extrinsic curvature* is defined as

$$K_{ab} = {h_a}^c{h_b}^d \nabla_c n_d.$$

Note that $K_{ab} = K_{(ab)}$, since we are assuming that the distribution induced by $\boldsymbol{n}$ is integrable in order for $\mathcal{M}$ to be foliated by $\boldsymbol{n}-$orthogonal hypersurfaces —recall that, by Frobenius' Theorem, integrability is equivalent to vanishing of the *twist*, $K_{[ab]}$. The extrinsic curvature can also be readily verified to be spatial. The extrinsic curvature relates the Levi-Civita connections of $\boldsymbol{g}$, $\boldsymbol{h}$ as follows

$$D_a v_b = {h_a}^c \nabla_c v_b + K_{ca}v^c n_b$$

for $v_a \in T\mathcal{S}$. The intrinsic and extrinsic curvatures of $\mathcal{S}$ are related to the curvature of $\mathcal{M}$ via the *Gauss–Codazzi* and *Codazzi–Mainardi* equations

$$r_{abcd} + K_{ac}K_{bd} - K_{ad}K_{bc} = {h_a}^p{h_b}^q{h_c}^r{h_d}^s R_{pqrs}, \tag{2.1.5a}$$

$$D_b K_{ac} - D_a K_{bc} = {h_a}^p{h_b}^q{h_c}^r n^s R_{pqrs}, \tag{2.1.5b}$$

on $\mathcal{S}$ —see [40], for instance. Since (2.1.5a) has all the symmetries of the Riemann tensor, all of the non-trivial components of (2.1.5a) are encoded in the trace part (with respect to $\boldsymbol{h}$):

$$r_{ac} + KK_{ac} - K_{ad}{K_c}^d = {h_a}^p{h_c}^r h^{qs} R_{pqrs}, \tag{2.1.6}$$

in accordance with the 3-dimensional Kulkarni–Nomizu decomposition. If the vacuum Einstein field equations are satisfied, one has $L_{ab} = (\lambda/6)g_{ab}$ and the 4−dimensional Kulkarni–Nomizu decomposition implies that

$$R_{abcd} = C_{abcd} + \tfrac{2\lambda}{3} g_{a[c}g_{d]b}.$$

Substituting into (2.1.5a)–(2.1.5b) yields

$$r_{ac} + KK_{ac} - K_{ad}K_c{}^d = S_{ac} + \tfrac{2\lambda}{3} h_{ac}, \tag{2.1.7a}$$

$$D_b K_{ac} - D_a K_{bc} = S_{cab}, \tag{2.1.7b}$$

where $S_{ac} \equiv n^d n^d C_{abcd}$ is the *electric part* of the Weyl curvature and

$$S_{cab} = S_{c[ab]} \equiv h_a{}^p h_b{}^q h_c{}^r n^s C_{pqrs}.$$

—see the next section for more details on the decomposition of "Weyl candidates". It is easily verified that S_{ab}, S_{abc} are tracefree with respect to $\boldsymbol{h}$ and spatial. Together, S_{ab} and S_{cab} completely determine C_{abcd} —see the forthcoming section for more details. Tracing (2.1.7a)–(2.1.7b), one obtains precisely the vacuum Einstein constraint equations (1.2.1a)–(1.2.1b).

Throughout this thesis we will restrict attention to orientable manifolds. In particular, $\boldsymbol{g}$ can be assumed to admit a *volume form* $\varepsilon_{abcd} = \epsilon_{[abcd]}$, which we normalise so that $\epsilon_{abcd}\epsilon^{abcd} = -24$. The induced volume form on $\mathcal{S}$ is given by the relation $\epsilon_{abc} = -n^d \epsilon_{abcd}$, which is reversed to give $\epsilon_{abcd} = 4n_{[a}\epsilon_{bcd]}$. The following identities (see see [41], for instance) will prove useful

$$\epsilon_{abcd}\epsilon^{pqrs} = -24\delta_{[a}{}^p\delta_b{}^q\delta_c{}^r\delta_{d]}{}^s, \tag{2.1.8a}$$

$$\epsilon_{ijk}\epsilon^{lmn} = 6\delta_{[i}{}^l\delta_j{}^m\delta_{k]}{}^n. \tag{2.1.8b}$$

2.1.3 Tensor spaces and their decompositions

Here we define some tensor spaces that will be used throughout this thesis:

- $\mathscr{C}(\mathcal{S})$, the space of scalar functions on $\mathcal{S}$;
- $\Lambda^1(\mathcal{S})$, the space of covectors over $\mathcal{S}$;
- $\mathscr{S}^2(\mathcal{S})$, the space of symmetric 2-tensors over $\mathcal{S}$;
- $\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$, the space of symmetric 2-tensors over $\mathcal{S}$ that are tracefree with respect to $\boldsymbol{h}$;
- $\mathscr{S}_{TT}(\mathcal{S};\boldsymbol{h})$, the space of symmetric 2-tensors over $\mathcal{S}$ that are *tracefree-transverse*[1] (or *TT*) with respect to $\boldsymbol{h}$;
- $\mathcal{J}(\mathcal{S})$, the space of *Jacobi* tensors —i.e. tensors J_{ijk} satisfying

$$J_{ijk} = J_{[ij]k}, \qquad J_{[ijk]} = 0.$$

The following lemma will be useful

[1]Recall that a 2-tensor η_{ij} is *transverse* if is has vanishing divergence: $D^i\eta_{ij} = 0$.

Lemma 1. Let $(\mathcal{S}, \boldsymbol{h})$ be a 3−dimensional Riemannian manifold, then there exists a cononical isomorphism between the spaces $\mathcal{J}(\mathcal{S})$ and $\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h}) \oplus \Lambda^1(\mathcal{S})$. More specifically, any given $J_{ijk} \in \mathcal{J}(\mathcal{S})$ has a unique decomposition of the form

$$J_{ijk} = \tfrac{1}{2}\left(\epsilon_{ij}{}^{l}F_{lk} + A_i h_{jk} - A_j h_{ik}\right), \tag{2.1.9}$$

with $A_i \in \Lambda^1(\mathcal{S})$ and $F_{ij} \in \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$. The fields $A_i,\ F_{ij}$ are given by

$$A_j \equiv J_{jk}{}^{k}, \qquad F_{km} \equiv \epsilon_{ij(m}J^{ij}{}_{k)}. \tag{2.1.10}$$

Proof. Given $A_i \in \Lambda^1(\mathcal{S})$ and $F_{ij} \in \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$, first note that J_{ijk}, as given in (2.1.9), does in fact define a Jacobi tensor. Indeed, it is clear that $J_{ijk} = J_{[ij]k}$, while $J_{[ijk]} = 0$ follows from contraction with the volume form:

$$\epsilon^{ijk}J_{ijk} = \tfrac{1}{2}\epsilon^{ijk}\left(\epsilon_{ij}{}^{l}F_{lk} + A_i h_{jk} - A_j h_{ik}\right) = \tfrac{1}{2}\epsilon^{ijk}\epsilon_{ij}{}^{l}F_{lk} = h^{lk}F_{lk} = 0,$$

where we have used (2.1.8b) and the fact that F_{ij} is tracefree. Expressions (2.1.10) follow by a similar calculation, and moreover imply injectivity of the map

$$(A_i, F_{ij}) \mapsto J_{ijk} \equiv \tfrac{1}{2}\left(\epsilon_{ij}{}^{l}F_{lk} + A_i h_{jk} - A_j h_{ik}\right).$$

It is straightforward to verify that at each $p \in \mathcal{S}$, $\Lambda^1(\mathcal{S}) \oplus \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$ and $\mathcal{J}(\mathcal{S})$ are both 8−dimensional as vector spaces. Hence, the above map is an isomorphism between $\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h}) \oplus \Lambda^1(\mathcal{S})$ and $\mathcal{J}(\mathcal{S})$. ☐

We will also require the electromagnetic decomposition of *Weyl candidates* on $(\mathcal{M}, \boldsymbol{g})$ —recall that a Weyl candidate is a tracefree 4−tensor, W_{abcd}, satisfying

$$W_{abcd} = -W_{bacd} = -W_{abdc}, \qquad W_{abcd} = W_{cdab}, \qquad W_{a[bcd]} = 0.$$

The Weyl tensor, C_{abcd}, is of course an example.

Electromagnetic decomposition of Weyl candidates

Let W_{abcd} be a Weyl candidate and define the *Hodge dual* as

$$W^{*}_{abcd} = \tfrac{1}{2}\epsilon_{cd}{}^{fg}W_{abcd}.$$

The *electric* and *magnetic* parts are then defined by

$$E_{ab} \equiv W_{acbd}n^c n^d, \qquad H_{ab} \equiv W^{*}_{acbd}n^c n^d.$$

Note that $E_{ab},\ H_{ab}$ are symmetric and intrinsic to $\mathcal{S}$, since $n^a E_{ab} = n^a n^c W_{acbd} n^d = 0$ (similarly for H_{ab}). They are moreover tracefree with respect to $\boldsymbol{h}$, as a consequence of W_{abcd} being tracefree with respect to $\boldsymbol{g}$. It is convenient to also define

$$B_{abc} \equiv h_a{}^{d}h_b{}^{f}h_c{}^{g}n^e W_{defg}.$$

It is straightforward to see that B_{abc} is intrinsic to $\mathcal{S}$, and is in fact a Jacobi tensor —i.e. an element of $\mathcal{J}(\mathcal{S})$. Now,

$$\begin{aligned} B_{abc} = h_a{}^d h_b{}^f h_c{}^g n^e W_{defg} &= (g_a{}^d + n_a n^d)(g_b{}^f + n_b n^f)(g_c{}^g + n_c n^g) n^e W_{defg} \\ &= n^d W_{adbc} - n_b E_{ac} + n_c E_{ab}, \end{aligned}$$

from which it follows that

$$H_{ae} \equiv \tfrac{1}{2}\epsilon_{ef}{}^{cd} n^b n^f C_{abcd} = \tfrac{1}{2}\epsilon_{ecdf} B_a{}^{df} n^c = -\tfrac{1}{2}\epsilon_{dfe} B_a{}^{df}.$$

Noting that B_{abc} is also tracefree with respect to $\boldsymbol{h}$, we see that this is just a special case of the decomposition of Jacobi tensors given in Lemma 1.

Defining $\zeta_{ab} \equiv h_{ab} + n_a n_b$, the Weyl candidate is given in terms of the electric and magnetic parts as follows

$$W_{abcd} = (\boldsymbol{\zeta} \owedge \boldsymbol{E})_{abcd} - 2\epsilon_{cd}{}^f n_{[a} H_{b]f} - 2\epsilon_{ab}{}^f n_{[c} H_{d]f}.$$

For a full derivation of this formula see [34].

2.2 Conformal geometry

In this section we describe some basic notions of conformal geometry and provide some transformation formulae that will be useful for later.

Recall that two pseudo-Riemannian manifolds $(\mathcal{M}, \boldsymbol{g})$ and $(\bar{\mathcal{M}}, \bar{\boldsymbol{g}})$ are said to be *conformally related* if there exists a diffeomorphism $\varphi : \bar{\mathcal{M}} \to \mathcal{M}$ and a positive function $\Theta : \mathcal{M} \to \mathbb{R}$ such that $\varphi^* \boldsymbol{g} = \Theta^2 \bar{\boldsymbol{g}}$. In particular, when $\mathcal{M} = \bar{\mathcal{M}}$, $\boldsymbol{g}$ and $\bar{\boldsymbol{g}}$ are said to be *conformally equivalent*. The Levi-Civita connections are then related by

$$(\nabla_a - \bar{\nabla}_a) v^c = \Upsilon_a{}^c{}_b v^b,$$

where $\Upsilon_a{}^c{}_b$ is the *transition tensor*, defined as

$$\Upsilon_a{}^c{}_b = (\delta_a{}^c \delta_b{}^d + \delta_a{}^d \delta_b{}^c - g_{ab} g^{cd}) d(\ln\Theta)_d \equiv (\delta_a{}^c \delta_b{}^d + \delta_a{}^d \delta_b{}^c - \bar{g}_{ab} \bar{g}^{cd}) d(\ln\Theta)_d.$$

See [34] for a derivation.

Definition 3. An n-dimensional Lorentzian (Riemannian) manifold $(\mathcal{M}, \boldsymbol{g})$ is (*locally*) *conformally flat* if, for each point $p \in \mathcal{M}$, there exists an open neighbourhood $\mathcal{U}$ for which $(\mathcal{U}, \boldsymbol{g})$ is conformally equivalent to an open neighbourhood of $\mathbb{R}^n$ equipped with the Minkowski (Euclidean) metric.

In dimension 4, conformal flatness is equivalent to vanishing of the Weyl tensor, while in dimension 3 it is equivalent to vanishing of the Cotton tensor (c.f. the *Weyl–Schouten Theorem*) which is defined as

$$Y_{ijk} \equiv D_i l_{jk} - D_j l_{ik}.$$

It is straightforward to verify that $Y_{ijk} \in \mathcal{J}(\mathcal{S})$. Moreover, note that

$$Y_{ij}{}^i = D^i l_{ij} - D_j l_i{}^i = D^j r_{ij} - \tfrac{1}{2} D_i r = 0,$$

by the contracted second Bianchi identity. Hence, Y_{ijk} is tracefree, and Lemma 1 implies that its essential components are encoded in the *Cotton–York* tensor, defined by

$$\mathcal{H}_{ij} \equiv \tfrac{1}{2}\epsilon_{km(i}Y^{km}{}_{j)}. \tag{2.2.1}$$

The following identity,

$$D_{[l}Y_{ij]k} = 0, \tag{2.2.2}$$

sometimes referred to as the *third Bianchi identity*, follows immediately from the definition of Y_{ijk}. Contracting the anti-symmetrised indices with the volume form, one obtains an equivalent identity in terms of the Cotton–York tensor,

$$D^i\mathcal{H}_{ij} = 0. \tag{2.2.3}$$

In addition to being transverse (i.e. divergence-free), the Cotton–York tensor is clearly tracefree. Hence, $\mathcal{H}_{ij} \in \mathscr{S}_{TT}(\mathcal{S};\boldsymbol{h})$.

2.2.1 Conformal transformation formulae

In dimensions 3 and 4, we have the following conformal transformation formulae (see [34]) for the Ricci curvature

$$\mathrm{Ric}[\boldsymbol{h}]_{ij} - \mathrm{Ric}[\Omega^{-2}\boldsymbol{h}]_{ij} = -\Omega^{-1}D_iD_j\Omega - h_{ij}h^{kl}(\Omega^{-1}D_kD_l\Omega - 2\Omega^{-2}D_k\Omega D_l\Omega), \tag{2.2.4}$$

$$\mathrm{Ric}[\boldsymbol{g}]_{ab} - \mathrm{Ric}[\Theta^{-2}\boldsymbol{g}]_{ab} = -2\Theta^{-1}\nabla_a\nabla_b\Theta - g_{ab}g^{cd}(\Theta^{-1}\nabla_c\nabla_d\Theta - 3\Theta^{-2}\nabla_c\Theta\nabla_d\Theta). \tag{2.2.5}$$

Taking the appropriate traces, one obtains the following transformation formulae for the scalar curvature

$$r[\boldsymbol{h}] - \Omega^{-2}r[\Omega^{-2}\boldsymbol{h}] = -4\Omega^{-1}\Delta_{\boldsymbol{h}}\Omega + 6\Omega^{-2}h^{ij}D_i\Omega D_j\Omega, \tag{2.2.6}$$

$$R[\boldsymbol{g}] - \Theta^{-2}R[\Theta^{-2}\boldsymbol{g}] = -6\Theta^{-1}\Box_{\boldsymbol{g}}\Theta + 12\Theta^{-2}g^{ab}\nabla_a\Theta\nabla_b\Theta, \tag{2.2.7}$$

where $\Delta_{\boldsymbol{h}} \equiv h^{ij}D_iD_j$ is the rough Laplacian, and $\Box_{\boldsymbol{g}} \equiv g^{ab}\nabla_a\nabla_b$ is the D'Alembertian. Alternatively, one can use the transformation law for the Schouten tensor

$$l[\boldsymbol{h}]_{ij} - l[\Omega^{-2}\boldsymbol{h}]_{ij} = -\Omega^{-1}D_iD_j\Omega + \tfrac{1}{2}\Omega^{-2}h^{kl}D_k\Omega D_l\Omega h_{ij}, \tag{2.2.8}$$

$$L[\boldsymbol{g}]_{ab} - L[\Theta^{-2}\boldsymbol{g}]_{ab} = -\Theta^{-1}\nabla_a\nabla_b\Theta + \tfrac{1}{2}\Theta^{-2}g^{cd}\nabla_c\Theta\nabla_d\Theta g_{ab}, \tag{2.2.9}$$

which is dimension-independent. In dimension 4, the remaining curvature components are contained in the Weyl tensor, which in $(1,3)$ form is conformally covariant:

$$C[\boldsymbol{g}]^a{}_{bcd} = C[\Theta^{-2}\boldsymbol{g}]^a{}_{bcd}.$$

If $\boldsymbol{g} = \Theta^2\bar{\boldsymbol{g}}$ on $\mathcal{M}$, then the induced metrics on a given hypersurface $\mathcal{S}$ are related by $\boldsymbol{h} = \theta^2\bar{\boldsymbol{h}}$, where $\theta \equiv \Theta|_{\mathcal{S}}$. Noting that the unit co-normals are related by $\bar{n}_a = \Theta^{-1}n_a$, it is straightforward to show that the extrinsic curvatures of $\mathcal{S} \hookrightarrow \mathcal{M}$ with respect to $\boldsymbol{g}$ and $\bar{\boldsymbol{g}}$ are related by

$$K_{ij} = \theta\left(\bar{K}_{ij} + \phi\bar{h}_{ij}\right), \tag{2.2.10}$$

where $\phi \equiv (g^{ab}n_a\nabla_b\Theta)|_{\mathcal{S}}$. See [34] for more details.

2.3 Functional Analysis

Here we collect some relevant results from elliptic PDE theory and functional analysis. The results are not necessarily stated in their most general form; rather, we present the forms most convenient for this thesis. Many of the results can be found in various forms in the Appendix of [42], Chapter 3 of [43] and Appendix II of [14], for instance. For an excellent account of some of the foundational results of PDE theory, with applications to General Relativity, see [44]. We will restrict the discussion here to second-order elliptic operators over closed Riemannian manifolds; the generalisations to mixed-order *Douglis–Nirenberg* systems and to open manifolds will be presented in the relevant chapters, namely Chapters 6 and 7.

The main tool we will use for proving the existence of solutions to the PDEs of interest is the Implicit Function Theorem, from now on the *IFT* —see [45], for example— which we state here for completeness.

Theorem. (Implicit Function Theorem) Let $\mathcal{X},\ \mathcal{Y},\ \mathcal{Z}$ be Banach spaces, and

$$\Psi : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$$

a mapping with continuous Fréchet derivative $D\Psi$. Suppose that $(x_0, y_0) \in \mathcal{X}\times\mathcal{Y}$ satisfies $\Psi(x_0, y_0) = 0$ and that the map $y \mapsto D\Psi(x_0, y_0)(0, y)$ is a Banach space isomorphism between $\mathcal{Y}$ and $\mathcal{Z}$. Then, there exist open neighbourhoods $\mathcal{U}$ of x_0 and $\mathcal{V}$ of y_0 and a Fréchet-differentiable mapping $\nu : \mathcal{U} \to \mathcal{V}$ such that $\Psi(x, \nu(x)) = 0$ for all $x \in \mathcal{U}$, and $\Psi(x, y) = 0$ for $(x, y) \in \mathcal{U} \times \mathcal{V}$ if and only if $y = \nu(x)$. Moreover, if the map $x \mapsto D\Psi(x_0, y_0)(x, 0)$ is injective, then ν is also injective.

2.3.1 Differential operators and notions of ellipticity

Let E, F be bundles over an n-dimensional manifold $\mathcal{S}$ and let P be a linear differential operator of order m between smooth sections, denoted $\Gamma(E),\ \Gamma(F)$, of the bundles E and F. Given $p \in \mathcal{S}$, there exists an open neighbourhood U and local coordinates x^α such that $E,\ F$ have local trivialisations $E\big|_U \simeq U \times \mathbb{R}^M,\ F\big|_U \simeq U \times \mathbb{R}^N$, with respect to which P may be written in the form

$$P = \sum_{|\alpha| \leq m} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

for some $A^\alpha \in \mathrm{Hom}(E, F)$, where $\alpha = (\alpha_1 \alpha_2 \cdots \alpha_n)$ is a *multi-index* and where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Definition 4. The *principal part* of P is given by

$$\sum_{|\alpha| = m} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

Given $\xi_i \in \Lambda^1(\mathcal{S})$, the *principal symbol of* of P *in the direction of* ξ_i, is the bundle homomorphism $\sigma_\xi[P] : E \to F$ given by

$$\sigma_\xi[P] = \sum_{|\alpha| = m} \xi_\alpha A^\alpha.$$

See [43] for an argument that $\sigma_\xi[P]$ is indeed independent of the choice of local trivialisation.

Throughout this thesis we will require the following notions of ellipticity:

Definition 5. A linear differential operator P is said to be *overdetermined elliptic* (resp. *underdetermined elliptic*) if at each $p \in \mathcal{S}$ and for each $\xi \in \Lambda^1_p(\mathcal{S})$, the symbol map $\sigma_\xi[P]$ is injective (resp. surjective). If P is both underdetermined and overdetermined elliptic —i.e. if $\sigma_\xi[P]$ is an isomorphism at each $p \in \mathcal{S}$— then we say that P is *(determined) elliptic.*

In Chapter 6 we will also require the weaker notion of *Douglis–Nirenberg ellipticity*; this is saved until later. The following lemma describes the behaviour of the principal symbol under composition of operators and the action the action of taking the adjoint —recall that the formal L^2-adjoint, P^*, is defined by

$$\langle P(\boldsymbol{u}), \boldsymbol{v}\rangle_{L^2(F)} = \langle \boldsymbol{u}, P^*(\boldsymbol{v})\rangle_{L^2(E)},$$

for arbitrary $\boldsymbol{u} \in \Gamma(E)$ and $\boldsymbol{v} \in \Gamma(F)$.

Lemma 2. Given two linear differential operators, $P_1 : \Gamma(E) \to \Gamma(F)$, $P_2 : \Gamma(F) \to \Gamma(G)$, of degrees $m_1,\ m_2$, then $P_2 \circ P_1$ has principal symbol

$$\sigma_\xi[P_2 \circ P_1] = \sigma_\xi[P_2] \circ \sigma_\xi[P_1]$$

and is of order at most $m_1 + m_2$. Moreover,

$$\sigma_\xi[P^*] = (\sigma_\xi[P])^*.$$

It follows that

(i) a linear differential operator P is overdetermined elliptic if and only if its formal adjoint, P^*, is underdetermined elliptic;

(ii) if P is overdetermined elliptic then $P^* \circ P$ is determined elliptic.

2.3.2 Sobolev spaces and elements of elliptic PDE theory

We begin by introducing some norms. Let $\|\cdot\|$ denote the pointwise norm, defined on an arbitrary (s,t)-tensor, $T_{i_1\cdots i_s}{}^{j_1\cdots j_t}$, as

$$\|\boldsymbol{T}\|^2 = h_{j_1 l_l}\cdots h_{j_t l_t} h^{i_1 k_l}\cdots h^{i_s k_s} T_{i_1\cdots i_s}{}^{j_1\cdots j_t} T_{k_1\cdots k_s}{}^{l_1\cdots l_t}.$$

For $k \geq 0$, let $\boldsymbol{u}$ be a $k-$times differentiable section of a bundle E. The *Sobolev k,p-norm* for $1 \leq p < \infty$ is defined by

$$\|\boldsymbol{u}\|_{k,p} = \left(\sum_{j=0}^{k}\int_{\mathcal{S}} \|\overbrace{DD\cdots D}^{j}\boldsymbol{u}\|^p\ d\mu\right)^{1/p}.$$

For compact manifolds, changing the metric and/or connection results in an equivalent norm. The completion of $\Gamma(E)$ with respect to $\|\cdot\|_{k,p}$ is the Sobolev space $W^{k,p}(E)$ —we will often omit reference to the bundle, for notational convenience. Of particular relevance for this thesis is the case $p = 2$; the norm will be denoted $\|\cdot\|_{H^k}$ and the Sobolev space by $H^k(E)$.

The Sobolev spaces admit a continuous multiplication property. In particular, we have the following (see Proposition 2.3 of [14], for instance[2]):

[2]The result is written there for Sobolev spaces on an open set $U \subset \mathbb{R}^n$ satisfying the *cone property*, but it extends

Proposition. (Schauder ring property) Let $k_1,\ k_2,\ k$ be non-negative integers satisfying

$$k_1 + k_2 > k + n/2 \qquad \text{and} \qquad k_1, k_2 \geq k.$$

Then the map $(\boldsymbol{u}, \boldsymbol{v}) \mapsto \boldsymbol{u} \otimes \boldsymbol{v}$ is a continuous mapping between $H^{k_1} \times H^{k_2}$ and H^k —that is to say, there exists a constant C such that, for all $\boldsymbol{u},\ \boldsymbol{v}$,

$$\|\boldsymbol{u} \otimes \boldsymbol{v}\|_{H^k} \leq C \|\boldsymbol{u}\|_{H^{k_1}} \|\boldsymbol{v}\|_{H^{k_2}}.$$

In particular, H^k is a Banach algebra (i.e. is closed under multiplication) for $k > n/2$.

Define also, for any non-negative integer k, the Banach space $C^k(E)$ consisting of $k-$differentiable sections of the bundle E, equipped with the norm

$$\|\boldsymbol{u}\|_{C^k} = \sup_{p \in \mathcal{S}} \sum_{|\alpha| \leq k} |D^\alpha \boldsymbol{u}|^2,$$

with α again denoting a multi-index. The spaces C^k and H^k are related via the *Sobolev embedding theorem*:

Theorem. (Sobolev embedding) Let $(\mathcal{S}, \boldsymbol{h})$ be a closed, smooth, $n-$dimensional Riemannian manifold. For each real number $s > k + n/2$, there is a constant C_s such that for all $\boldsymbol{u} \in H^s$

$$\|\boldsymbol{u}\|_{C^k} \leq C_s \|\boldsymbol{u}\|_{H^s}$$

—i.e. there is a continuous embedding $H^s \subset C^k$.

See Theorem 2.5 of [43] or Corollary 2.2 in Appendix I of [14], for instance. The following embedding theorem will also prove useful:

Theorem. (Rellich–Kondrakov) Let $(\mathcal{S}, \boldsymbol{h})$ be a closed $n-$dimensional Riemannian manifold then there is a compact embedding

$$W^{k,p} \subset W^{l,q}$$

for $k > l$ and $k - n/p > l - n/q$. That is to say, every bounded sequence in $W^{k,p}$ has a Cauchy subsequence in $W^{l,q}$.

See Theorem 2.6, Chapter III of [43], or Theorem 6 of Appendix C in [42]. The most important consequence for our purposes (taking $p = q = 2$) is that the inclusion $H^k \subset H^l$ is compact for $k > l$. For non-negative integers k, we also define the sub-Banach spaces of *mean-zero* functions

$$\bar{H}^k(\mathscr{C}(\mathcal{S}); \boldsymbol{h}) \equiv \left\{ f \in H^k(\mathscr{C}(\mathcal{S})) \;\middle|\; \int_{\mathcal{S}} f \; d\mu_{\boldsymbol{h}} = 0 \right\}.$$

The fact that $\bar{H}^k(\mathscr{C}(\mathcal{S}); \boldsymbol{h})$ is indeed a closed subspace of $H^k(\mathscr{C}(\mathcal{S}))$ is a result of the continuous embedding $H^k \subset L^1$ when $k \geq 0$ and the fact that $\mathcal{S}$ is closed —the $k = 0$ case follows from Cauchy Schwarz, while the $k > 0$ cases follow from the Rellich–Kondrakov theorem with $l = 0, p = 2, q = 1$.

It will convenient to collect here some of the basics of elliptic PDE theory. First note that a differential operator $P : \Gamma(E) \to \Gamma(F)$ of order m extends to a bounded linear map $P : H^{k+m} \to H^k$

to Sobolev spaces over smooth Riemannian manifolds, as remarked in Section 2.2 of Appendix I.

for all $k \geq 0$. If P is an elliptic operator of order m, with smooth coefficients, and if $\boldsymbol{u} \in H^k$ $(k \geq m)$ is a solution of

$$P(\boldsymbol{u}) = \boldsymbol{f},$$

with $\boldsymbol{f} \in H^k$, then in fact $\boldsymbol{u} \in H^{k+m}$. Moreover, there exists a constant $C_k > 0$ such that for all such solutions, $\boldsymbol{u} \in H^k$,

$$\|\boldsymbol{u}\|_{H^{k+m}} \leq C_k \left(\|\boldsymbol{f}\|_{H^k} + \|\boldsymbol{u}\|_{H^{k+m-1}} \right). \tag{2.3.1}$$

—see [46], for example. As an immediate consequence, one has *elliptic regularity*: a solution $\boldsymbol{u} \in H^m$ of $P(\boldsymbol{u}) = 0$ is necessarily smooth. The estimate (2.3.1), which is the basis of the *Fredholm theory* of elliptic operators, is sometimes called the "fundamental elliptic estimate".

Definition 6. Let $\mathcal{B}_1$, $\mathcal{B}_2$ be Banach spaces. The *cokernel* of an operator $T : \mathcal{B}_1 \to \mathcal{B}_2$ is the quotient coker $T \equiv \mathcal{B}_2/\overline{T(\mathcal{B}_1)}$. The operator T is said to be *Fredholm* if both the kernel and cokernel are finite dimensional and if the image of T is closed in $\mathcal{B}_2$.

Corollary. (Fredholm alternative) Let E, F be bundles over a closed Riemannian manifold $(\mathcal{S}, \boldsymbol{h})$ and let $P : H^{s+m}(E) \to H^s(F)$, $s \geq m$ be a linear elliptic operator of order m with smooth coefficients. Then, the kernel and cokernel of P are finite-dimensional, closed, subspaces and

$$H^s(F) = \mathrm{Im}\ P|_{H^{s+m}(E)} \oplus \ker\ P^*|_{H^s(F)},$$

where the subspaces are L^2-orthogonal and each is closed in $H^s(F)$.

Consequently, for $P : H^{s+m}(E) \to H^s(F)$ to be surjective, it is necessary and sufficient for P^* to be injective on $H^s(F)$.

The result is generalised to operators that are either overdetermined or underdetermined elliptic, and of arbitrary order, in the following *Splitting Lemma*:

Lemma 3. Let E, F be bundles over a closed Riemannian manifold $(\mathcal{S}, \boldsymbol{h})$ and let

$$\mathscr{D} : H^{s+k}(E) \longrightarrow H^s(F), \qquad s \geq k,$$

be a differential operator of order k with smooth coefficients, and $\mathscr{D}^*$ the formal L^2-adjoint. Suppose that $\mathscr{D}$ is either overdetermined elliptic or underdetermined elliptic, then

$$H^s(F) = Im\ \mathscr{D}|_{H^{s+k}(E)} \oplus ker\ \mathscr{D}^*|_{H^s(F)},$$

and

$$H^s(E) = Im\ \mathscr{D}^*|_{H^{s+k}(F)} \oplus ker\ \mathscr{D}|_{H^s(E)},$$

where (in each case) both factors are closed and are L^2-orthogonal. In particular, if $\mathscr{D}|_{H^{s+2k}(E)}$ is injective, then $\mathscr{D}^* : H^{s+k}(F) \to H^s(E)$ is surjective, and the composition $\mathscr{D}^* \circ \mathscr{D} : H^{s+2k}(E) \to H^s(F)$ is an isomorphism.

See Corollary 32 in Appendix *I* of [42], or Lemma 2.3 in Chapter VII of [14], for instance. The proof proceeds by applying the Fredholm alternative to the elliptic operator $\mathscr{D}^* \circ \mathscr{D}$, for $\mathscr{D}$ overdetermined elliptic, or to $\mathscr{D} \circ \mathscr{D}^*$ for $\mathscr{D}$ underdetermined elliptic. An immediate consequence of the splitting lemma is the so-called *York split*: any $\eta_{ij} \in H^s(\mathscr{S}^2_0(\mathcal{S}; \boldsymbol{h}))$, $s \geq 1$, can be decomposed as

$$\eta_{ij} = L(\boldsymbol{X})_{ij} + T_{ij}, \tag{2.3.2}$$

for some $X_i \in H^{s+1}(\Lambda^1(\mathcal{S}))$ and some $T_{ij} \in H^s(\mathscr{S}^2_{TT}(\mathcal{S};\boldsymbol{h}))$, where here L denotes the *conformal Killing operator*, which is the formal adjoint of the divergence operator restricted to $\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$ —see Section 4.1.1.

We also mention the following special case of Theorem 2.1 in Appendix II of [14], which can be considered a partial generalisation of some of the results discussed above (which were all stated for elliptic operators P with smooth coefficients).

Theorem 1. Let $(\mathcal{S},\boldsymbol{h})$ be a smooth closed 3-dimensional Riemannian manifold and let

$$P \equiv \sum_{|\alpha|\leq 2} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

be a second-order elliptic operator between bundles E and F over $\mathcal{S}$. Suppose that $A^\alpha \in H^2$ for $|\alpha| = 2$, $A^\alpha \in H^1$ for $|\alpha| = 1$ and $A^0 \in L^2$. Then it follows that

(i) P is a continuous mapping $H^2(E) \to L^2(F)$;

(ii) The following estimate holds for all $\boldsymbol{u} \in H^2(E)$:

$$\|\boldsymbol{u}\|_{H^2} \leq C(\|P(\boldsymbol{u})\|_{L^2} + \|\boldsymbol{u}\|_{H^1}), \tag{2.3.3}$$

where the constant C depends only on the norms of the coefficients A^α and the *ellipticity constant* of P (see [14]).

In Chapter 6 we will discuss an analogue of the above theorem for so-called *Douglis–Nirenberg* elliptic operators (see Theorem 5), and in Chapter 8 a version of the above will be given for non-compact $\mathcal{S}$, in terms of weighted Sobolev spaces (see Propositions 26 and 27).

Chapter 3

The Conformal Constraint Equations

In this chapter we will introduce the system of equations of interest in this thesis, namely the *Conformal Constraint Equations* (or *CCEs*) of H. Friedrich, and reductions thereof. Much of the thesis will be concerned with a reduction of the CCEs called the *Extended Constraint Equations* (or *ECEs*), which will be introduced in Section 3.3.2. As we shall see, the ECEs are entirely equivalent to the Einstein constraint equations, but have more manifest structure. We will return to the full CCEs in Chapter 7.

In this chapter, we will begin with a brief discussion of the *Conformal Field Equations* (or *CFEs*), for which the CCEs are the constraint equations i.e. obtained via a $3+1$ decomposition. We will then discuss the equivalence of the CCEs with the Einstein constraint equations, and describe their behaviour under conformal transformations —what we will refer to as their *quasi conformal covariance*. In addition, we will give a system of integrability conditions which will prove important in subsequent sections.

Finally, we will describe various simplifications of the CCEs, including the ECEs. While the study of the ECEs is a necessary first step towards an understanding of the full CCEs, they are also of significant interest (as mentioned previously) as an alternative route for the construction of initial data, with explicit control on certain components of the 4-dimensional Weyl curvature.

3.1 Introducing the Conformal Field Equations

3.1.1 Asymptotic simplicity

In the seminal work [31], Penrose suggested a new way of studying the asymptotic properties of spacetimes, based on the notion of conformal extensions. Roughly speaking, *Penrose's proposal* states that in an isolated gravitating system, one should expect that the physical fields admit a smooth conformal extension similar to that of the Minkowski spacetime. Such a spacetime is modelled mathematically through the notion of *asymptotic simplicity.*

Definition 7. A spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is *asymptotically simple* if there exists a smooth, oriented, time-oriented, causal manifold $(\mathcal{M}, \boldsymbol{g})$ with boundary $\mathscr{I} \equiv \partial\mathcal{M}$ and a smooth function Ξ such that

(i) $\Xi > 0$ on $\mathcal{M} \setminus \mathscr{I}$, $\Xi = 0$ and $\boldsymbol{d}\Xi \neq 0$ on $\mathscr{I}$;

(ii) there exists an embedding $\varphi : \tilde{\mathcal{M}} \hookrightarrow \mathcal{M}$ such that $\varphi(\tilde{\mathcal{M}}) = \mathcal{M} \setminus \mathscr{I}$ and $\varphi^* \boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}$;

(iii) each null geodesic of $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ acquires two endpoints on $\mathscr{I}$.

The manifold $(\mathcal{M}, \boldsymbol{g})$ is referred to as the *unphysical spacetime* with *conformal boundary* $\mathscr{I}$, and can be considered a conformal extension of the *physical spacetime*, $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. Note that condition (i) ensures that $\mathscr{I}$ is a hypersurface of $\mathcal{M}$ and that $\mathscr{I}$ is at infinity from the perspective of the physical metric, $\tilde{\boldsymbol{g}}$.

The Minkowski, de Sitter and anti-de Sitter spacetimes, with line elements

$$
\begin{aligned}
&\tilde{\boldsymbol{\eta}} = -dt \otimes dt + r^2 dr \otimes dr + r^2 \boldsymbol{\sigma}, \\
&\tilde{\boldsymbol{g}}_{dS} = -dt \otimes dt + a^2 \cosh^2(t/a), \qquad a = \sqrt{3/\lambda}, \qquad \lambda > 0, \\
&\tilde{\boldsymbol{g}}_{adS} = -\cosh^2 r dt \otimes dt + a^2 (dr \otimes dr + \sinh^2 r \boldsymbol{\sigma}), \qquad a = \sqrt{3/|\lambda|}, \qquad \lambda < 0,
\end{aligned}
$$

are the prototypical examples of asymptotically-simple spacetimes. In fact, they can all be conformally extended by mapping into the Einstein cylinder

$$
\mathcal{M}_{\mathcal{E}} = \mathbb{R} \times \mathbb{S}^3, \qquad \boldsymbol{g}_{\mathcal{E}} = -dT \otimes dT + d\psi \otimes d\psi + \sin^2 \psi \boldsymbol{\sigma},
$$

where here $\boldsymbol{\sigma}$ denotes the standard metric on $\mathbb{S}^2$, $T \in (-\infty, \infty)$ and $\psi \in [0, 2\pi]$. On the other hand, condition (iii) is violated by the Schwarzschild spacetime, for example, which has line element

$$
ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \boldsymbol{\sigma},
$$

given here in Kruskal–Szekeres form. The Schwrazschild spacetime is, however, *weakly asymptotically simple*. That is to say that, its asymptotic region is diffeomorphic to that of an asymptotically simple spacetime —see [34] for a precise definition. The Nariai spacetime is an example of a smooth spacetime which fails to be even weakly asymptotically simple —see Chapter 7 of [34]. The question of the genericity of (weakly) asymptotically simple spacetimes is an important open quesion. The reader is again referred to [34] for further discussion.

3.1.2 The Conformal Einstein Field equations (CEFEs)

From the transformation formula of the Ricci curvature, we see that the vacuum Einstein field equations (with cosmological constant λ) are rewritten in terms of the conformal factor Ξ and the unphysical metric, $\boldsymbol{g} \equiv \Xi^2 \tilde{\boldsymbol{g}}$, as follows

$$
\begin{aligned}
0 &= \mathrm{Ric}[\tilde{\boldsymbol{g}}]_{ab} - \lambda \tilde{g}_{ab} \\
&= \mathrm{Ric}[\Xi^{-2} \boldsymbol{g}]_{ab} - \lambda \Xi^{-2} g_{ab} \\
&= \mathrm{Ric}[\boldsymbol{g}]_{ab} - \lambda \Xi^{-2} g_{ab} + 2\Xi^{-1} \nabla_a \nabla_b \Xi + \Xi^{-2} g_{ab} g^{cd} (\Xi \nabla_c \nabla_d \Xi - 3 \nabla_c \Xi \nabla_d \Xi).
\end{aligned}
$$

Hence, when expressed in terms of unphysical quantities, the Einstein field equations become singular as one approaches the conformal boundary, $\mathscr{I}$, on which $\Xi = 0$. In order to take full advantage of the notion of asymptotic simplicity, it is advantageous to have a conformally-regular version of the Einstein field equations. This is precisely what the *Conformal Field Equations* of H. Friedrich provide —see [32, 33].

For the sake of completeness, we include here a brief discussion of the CFEs. For a more detailed discussion, we refer the reader to [34]. The vacuum CFEs are given by the vanishing of the following *zero-quantities*

$$Z_{ab} \equiv \nabla_a \nabla_b \Xi + \Xi L_{ab} - s g_{ab}, \tag{3.1.1a}$$

$$Z_a \equiv \nabla_a s + L_{ac} \nabla^c \Xi, \tag{3.1.1b}$$

$$\Delta_{bac} \equiv \nabla_b L_{ac} - \nabla_a L_{bc} - d_{abcd} \nabla^d \Xi, \tag{3.1.1c}$$

$$\Lambda_{abc} \equiv \nabla_e d^e{}_{abc}, \tag{3.1.1d}$$

$$Z \equiv \lambda - 6\Xi s + 3\nabla_a \Xi \nabla^a \Xi. \tag{3.1.1e}$$

Here, Ξ is the conformal factor, L_{ab} is the Schouten tensor, which recall is defined in terms of the Ricci tensor R_{ab} and the Ricci scalar R via

$$L_{ab} = \frac{1}{2} R_{ab} - \frac{1}{12} R g_{ab}, \tag{3.1.2}$$

s is the so-called *Friedrich scalar* defined as

$$s \equiv \tfrac{1}{4} \nabla_a \nabla^a \Xi + \tfrac{1}{24} R \Xi \tag{3.1.3}$$

and $d^a{}_{bcd}$ denotes the *rescaled Weyl tensor*, defined as

$$d^a{}_{bcd} \equiv \Xi^{-1} C^a{}_{bcd}.$$

Recall that $C^a{}_{bcd}$ denotes the Weyl tensor. The geometric meaning of these zero-quantities is as follows: the equation $Z_{ab} = 0$ encodes the conformal transformation formula between R_{ab} and $\tilde{R}_{ab}$, and the equation $Z_a = 0$ is obtained as an integrability condition by considering $\nabla^a Z_{ab}$ and commuting covariant derivatives. Equations $\Delta_{abc} = 0$ and $\Lambda_{abc} = 0$ encode the second Bianchi identity. Finally, $Z = 0$ is a constraint in the sense that if it is verified at one point $p \in \mathcal{M}$ then $Z = 0$ holds in $\mathcal{M}$ (recalling that we are assuming $\mathcal{M}$ to be connected) by virtue of the equations $Z_{ab} = 0,\ Z_a,\ \Delta_{abc} = \Lambda_{abc} = 0$.

A solution to the CFEs consists of a collection of fields

$$(g_{ab},\ \Xi,\ s\ , L_{ab},\ d_{abcd})$$

satisfying

$$Z_{ab} = 0, \quad Z_a = 0, \quad \Delta_{abc} = 0, \quad \Lambda_{abc} = 0, \quad Z = 0,$$

on a manifold with boundary, $\mathcal{M}$. Given a solution to the CFEs, the physical metric $\tilde{\boldsymbol{g}} \equiv \Xi^{-2}\boldsymbol{g}$ solves the Einstein field equations on $\tilde{\mathcal{M}} \equiv \mathcal{M} \setminus \partial\mathcal{M}$.

An important feature of the CFEs is that they admit a hyperbolic reduction —that is to say that an *auxiliary system* of wave equations can be derived for the fields $g_{ab}, \Xi, s, L_{ab}, d_{abcd}$, which if satisfied imply a *subsidiary system* of wave equations for the resulting zero quantities $Z_{ab}, Z_a, \Delta_{abc}, \Lambda_{abc}, Z$. A uniqueness theorem for solutions of wave equations then ensures that if the zero quantities and their normal derivatives vanish on a given spacelike hypersurface, then they will continue to vanish on the spacetime development —i.e. the CFEs will be *propagated*.

3.2 Introducing the Conformal Constraint Equations

Note that, for the same reason as for the Einstein field equations, the conformally-rescaled version of the Einstein constraint equations (see (3.2.4a) and (3.2.4b), later) is bound to become degenerate wherever the hypersurface intersects the conformal boundary, $\mathscr{I}$. Hence, in order to study the Cauchy problem in the unphysical setting, one requires a system of conformally-regular constraint equations. This is precisely what the Conformal Constraint Equations (CCEs) provide.

The CCEs are the constraint equations implied by the CFEs on an embedded hypersurface $\mathcal{S} \hookrightarrow \mathcal{M}$, via a $3+1$ decomposition. They can be derived using the projector formalism of Section 2.1.2, or alternatively using frame formalism —for a full derivation, including matter sources, we refer the reader to [34]. We will consider here only the case of a spacelike hypersurface $\mathcal{S}$ in a vacuum unphysical spacetime, for which the fields of interest are as follows:

- Ω, the restriction of Ξ to $\mathcal{S}$,
- $\sigma \equiv (n^a \nabla_a \Xi)|_{\mathcal{S}}$, the normal derivative of Ξ restricted $\mathcal{S}$,
- s, the restriction of the Friedrich scalar to $\mathcal{S}$,
- L_i, the normal-tangential components of L_{ab}, given by $L_a \equiv h_a{}^b n^c L_{ab}$,
- L_{ij}, the tangential-tangential components of L_{ab}, given by $L_{ab} \equiv h_a{}^c h_b{}^d L_{cd}$,
- K_{ij}, the unphysical (i.e. with respect to $\boldsymbol{g}$) extrinsic curvature of $\mathcal{S} \hookrightarrow \mathcal{M}$,
- $d_{ij},\ d^*_{ij}$, the electric and magnetic parts of $d_{abcd} \equiv g_{af} d^f{}_{bcd}$ —see Section 2.1.3,
- h_{ij}, the unphysical induced Riemannian metric on $\mathcal{S}$.

Here, n^a denotes the unit normal to $\mathcal{S}$ in $\mathcal{M}$, with respect to $\boldsymbol{g}$. The CCEs on the hypersurface $\mathcal{S}$ are then given the vanishing of the zero quantities

$$P_{ij} = 0, \quad Z_i = 0, \quad W_i, \quad X_{ij} = 0, \quad Y_{ijk} = 0, \quad X_{ijk} = 0, \quad \Lambda^*_i = 0, \quad \Lambda_i = 0, \quad A = 0,$$

defined by

$$P_{ij} \equiv D_i D_j \Omega - \sigma K_{ij} - s h_{ij} + \Omega L_{ij}, \tag{3.2.1a}$$

$$Z_i \equiv D_i \sigma - K_i{}^k D_k \Omega + \Omega L_i, \tag{3.2.1b}$$

$$W_i \equiv D_i s + L_{ij} D^j \Omega - \sigma L_i, \tag{3.2.1c}$$

$$X_{ij} \equiv D_i L_j - D_j L_i + K_j{}^k L_{ik} - K_i{}^k L_{jk} - \epsilon_{ijl} d^{*}_k{}^l D^k \Omega, \tag{3.2.1d}$$

$$X_{ijk} \equiv D_i L_{jk} - D_j L_{ik} - 2K_{k[i} L_{j]} - \epsilon_{ijl} d^{*}_k{}^l \sigma + 2 d_{k[i} D_{j]} \Omega - 2 d_{l[i} h_{j]k} D^l \Omega, \tag{3.2.1e}$$

$$Y_{ijk} \equiv D_i K_{jk} - D_j K_{ik} - \Omega \epsilon_{ijl} d^{*}_k{}^l + h_{jk} L_i - h_{ik} L_j, \tag{3.2.1f}$$

$$\Lambda^*_i \equiv D^j d^*_{ij} - \epsilon_{ikl} K_j{}^l d^{jk}, \tag{3.2.1g}$$

$$\Lambda_i \equiv D^j d_{ij} + \epsilon_{ikl} K_j{}^l d^{*jk}, \tag{3.2.1h}$$

$$U_{ij} \equiv l_{ij} - L_{ij} - \Omega d_{ij} - K_i{}^k K_{jk} + \tfrac{1}{4} K_{kl} K^{kl} h_{ij} - \tfrac{1}{4} K^2 h_{ij} + K K_{ij}, \tag{3.2.1i}$$

$$A \equiv \lambda - 6\Omega s - 3\sigma^2 + 3 D_i \Omega D^i \Omega, \tag{3.2.1j}$$

Note that $Y_{ijk} = 0$ and $U_{ij} = 0$ are precisely the Codazzi Mainardi and Gauss Codazzi equations for the embedding $\mathcal{S} \hookrightarrow \mathcal{M}$ – see Section 2.1.2. All other equations are obtained via projections of the CFEs, see [34] for the full details. In the zero quantity U_{ij}, l_{ij} denotes the 3−dimensional Schouten curvature (see Section 2.1.1), which can be considered a second-order differential operator on the components of h_{ij}. The equation $U_{ij} = 0$ can therefore be read as a second-order differential equation for h_{ij}. Note that under the exchange

$$d_{ij} \longrightarrow d^*_{ij}, \qquad d^*_{ij} \longrightarrow -d_{ij},$$

the zero quantities Λ_i, Λ^*_i undergo the following transformation

$$\Lambda_i \longrightarrow \Lambda^*_j, \qquad \Lambda^*_i \longrightarrow -\Lambda_i,$$

which is analogous to the *electromagnetic duality* enjoyed by the Maxwell equations. For this reason, we will call the equations $\Lambda_i = \Lambda^*_i = 0$ the *electromagnetic constraints*. An important feature of the *algebraic constraint*, $A = 0$, is that it need only be imposed at a single point $p \in \mathcal{S}$, the other equations guarantee that it will be satisfied everywhere on $\mathcal{S}$. [1] More specifically, we have

$$D_j A = -6\Omega W_j + 6\left(D^i\Omega\right) P_{ij} - 6\sigma Z_j, \tag{3.2.2}$$

so that if $Z_i = W_i = 0$ and $P_{ij} = 0$ then A is necessarily constant on $\mathcal{S}$ and therefore vanishes everywhere if it vanishes at any single $p \in \mathcal{S}$. This is sometimes referred to as the *propagation of the cosmological constant*. The identity (3.2.2) is in fact one of a larger system of integrability conditions, which will be discussed in Section 3.2.3.

For use in Chapter 7, we define here the *conformal constraint map*

$$\begin{aligned}\Xi : &\mathscr{C}(\mathcal{S}) \times \mathscr{C}(\mathcal{S}) \times \mathscr{C}(\mathcal{S}) \times \Lambda^1(\mathcal{S}) \times \mathscr{S}^2(\mathcal{S}) \times \mathscr{S}^2(\mathcal{S}) \times \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h}) \times \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h}) \times \mathscr{S}^2(\mathcal{S}) \\ &\longrightarrow \mathscr{S}^2(\mathcal{S}) \times \Lambda^1(\mathcal{S}) \times \Lambda^1(\mathcal{S}) \times \Lambda^2(\mathcal{S}) \times \mathcal{J}(\mathcal{S}) \times \mathcal{J}(\mathcal{S}) \times \Lambda^1(\mathcal{S}) \times \Lambda^1(\mathcal{S}) \times \mathscr{C}(\mathcal{S}) \times \mathscr{S}^2(\mathcal{S}),\end{aligned} \tag{3.2.3}$$

given by

$$\Xi(\Omega, \sigma, s, L_i, L_{ij}, K_{ij}, d^*_{ij}, d_{ij}, h_{ij}) = (P_{ij}, Z_i, W_i, X_{ij}, Y_{ijk}, X_{ijk}, \Lambda^*_i, \Lambda_i, A).$$

3.2.1 Relationship to the Einstein constraint equations

A solution of the Einstein constraint equations naturally extends to a solution of the CCEs as follows: we first fix $\Omega = 1$, $\sigma = 0$, corresponding to trivial intrinsic and extrinsic conformal rescalings (consistent with $\Xi = 0$). Substituting into $A = 0$, one obtains $s = \lambda/6$; substituting into $P_{ij} = 0$ one finds $L_{ij} = (\lambda/6)h_{ij}$, and then $Z_i = 0$ implies $L_i = 0$, while $Y_{ijk} = 0$ implies

$$d^*_{ij} = \epsilon_{ikl} D^k K^l{}_j,$$

and $U_{ij} = 0$ implies

$$d_{ij} = l_{ij} - K_i{}^k K_{jk} - \tfrac{1}{6}\lambda h_{ij} + \tfrac{1}{4}K_{kl}K^{kl}h_{ij} - \tfrac{1}{4}K^2 h_{ij} + KK_{ij}.$$

At this point the only remaining equations are $\Lambda_i = \Lambda^*_i = 0$. One can check that these are indeed satisfied automatically by virtue of the other equations. More specifically, the zero quantities

[1] Recall that we are assuming that $\mathcal{S}$ is connected.

$\Lambda_i,\ \Lambda^*_i,\ Y_{ijk},\ X_{ij},\ X_{ijk},\ U_{ij}$ satisfy the following identities

$$\begin{aligned}
\epsilon_{ijk}D^k Y^{ij}{}_l &= -2\Omega\Lambda^*_l + \epsilon_{lij}X^{ij} - 2\epsilon_{ljk}K^{ij}U_i{}^k,\\
D^iU_{ij} - D_jU_i{}^i &= X_j{}^i{}_i - \Omega\Lambda_j + K_{ik}Y_j{}^{ik} - K_{jk}Y^{ik}{}_i - KY_j{}^i{}_i.
\end{aligned}$$

Setting $\Omega = 1$ along with $Y_{ijk} = X_{ijk} = 0,\ X_{ij} = U_{ij} = 0$ we indeed see that $\Lambda_i = \Lambda^*_i = 0$. These, along with identity (3.2.2) are part of a larger system of integrability relations, which are described in detail in Section 3.2.3.

Conversely, a solution of the CCEs determines a solution of the Einstein constraint equations. To see this, first note that one has the following identities

$$\begin{aligned}
\Omega^2 r - 6D_i\Omega D^i\Omega + 4\Omega D_iD^i\Omega - \Omega^2(K_{ij}K^{ij} - K^2) - 2\lambda - 4\Omega K\sigma + 6\sigma^2&\\
= 4\Omega P_i{}^i + 4\Omega^2 U_i{}^i - 2A,& \qquad (3.2.4a)\\
\Omega^3 D^i(\Omega^{-2}K_{ij}) - \Omega D_jK + 2D_j\sigma = -\Omega Y_j{}^i{}_i + 2Z_j.& \qquad (3.2.4b)
\end{aligned}$$

Assuming $P_{ij} = 0,\ Z_i = 0,\ Y_{ijk} = 0,\ U_{ij} = 0,\ A = 0$, the above imply the so-called *conformal Hamiltonian* and *conformal Momentum* constraints. Note that if one sets $\Omega = 1,\ \sigma = 0$ then they reduce to the standard Hamiltonian and Momentum constraints (1.2.1a) and (1.2.1b). More generally, if one defines

$$\tilde{h}_{ij} \equiv \Omega^{-2}h_{ij},\ \tilde{K}_{ij} \equiv \Omega^{-1}K_{ij} - \Omega^{-2}\sigma h_{ij}$$

and re-expresses (3.2.4a)–(3.2.4b) in terms of the physical metric $\tilde{h}_{ij}$ and physical curvature quantities, using the transformation formula (2.2.6) from Section 2.2, then one obtains precisely the Einstein constraint equations for $\tilde{h}_{ij},\ \tilde{K}_{ij}$. Note that the expression for $\tilde{K}_{ij}$ is obtained from the conformal transformation formula (2.2.10) with $\theta = \Omega$ and $\phi = \sigma$.

The above observations are collected in the following proposition:

Proposition 1. Given a solution

$$(\Omega, \sigma, s, L_i, L_{ij}, K_{ij}, d^*_{ij}, d_{ij}, h_{ij})$$

of the CCEs on $\mathcal{S}$, the fields

$$\tilde{h}_{ij} \equiv \Omega^{-2}h_{ij},\ \tilde{K}_{ij} \equiv \Omega^{-1}K_{ij} - \Omega^{-2}\sigma h_{ij} \qquad (3.2.5)$$

comprise a solution to the Einstein constraint equations, wherever $\Omega > 0$, with cosmological constant λ. Conversely an initial data set $\tilde{h}_{ij},\ \tilde{K}_{ij}$ may be canonically extended to a solution of the CCEs setting

$$\tilde{\Omega} = 1, \quad \tilde{\sigma} = 0, \quad \tilde{s} = \tfrac{\lambda}{6}, \quad \tilde{L}_i = 0, \quad \tilde{L}_{ij} = \tfrac{\lambda}{6}\tilde{h}_{ij}, \quad \tilde{d}^*_{im} = -\tilde{\epsilon}_{jk(m}\tilde{D}^k\tilde{K}_{i)}{}^j, \qquad (3.2.6)$$

$$d_{ij} = l_{ij} - K_i{}^kK_{jk} - \tfrac{1}{6}\lambda h_{ij} + \tfrac{1}{4}K_{kl}K^{kl}h_{ij} - \tfrac{1}{4}K^2h_{ij} + KK_{ij}. \qquad (3.2.7)$$

Note that the expression d^*_{ij} is tracefree automatically by symmetry considerations. The field d_{ij} is clearly symmetric and is tracefree by virtue of the Hamiltonian constraint. In the forthcoming section, we shall see that the CCEs are (quasi) conformally covariant, in the sense that they admit a conformal transformation which maps solutions to solutions. Hence, given a solution to the Einstein constraint equations, there is a corresponding family of conformally-related solutions of the CCEs.

3.2.2 Quasi conformal covariance

The CCEs enjoy a notion of conformal covariance: given smooth functions $\theta > 0$ and ϕ, then the following fields

$$\acute{\Omega} \equiv \theta\Omega, \tag{3.2.8a}$$
$$\acute{\sigma} \equiv \sigma + \Omega\phi, \tag{3.2.8b}$$
$$\acute{s} \equiv \theta^{-1}s - \tfrac{1}{2}\theta^{-1}\phi(\Omega\phi + 2\sigma) + \theta^{-2}D_i\theta D^i\Omega + \tfrac{1}{2}\theta^{-3}\Omega D_i\theta D^i\theta, \tag{3.2.8c}$$
$$\acute{L}_{ij} \equiv L_{ij} + K_{ij}\phi - \theta^{-1}D_iD_j\theta + 2\theta^{-2}D_i\theta D_j\theta + \tfrac{1}{2}\theta^{-2}h_{ij}(\theta^2\phi^2 - D_k\theta D^k\theta), \tag{3.2.8d}$$
$$\acute{L}_i \equiv \theta^{-1}L_i + \theta^{-2}\phi D_i\theta - \theta^{-1}D_i\phi + \theta^{-2}K_{ij}D^j\theta, \tag{3.2.8e}$$
$$\acute{K}_{ij} \equiv \theta(K_{ij} + \phi h_{ij}), \tag{3.2.8f}$$
$$\acute{d}_{ij} \equiv \theta^{-1}d_{ij}, \tag{3.2.8g}$$
$$\acute{d}^*_{ij} \equiv \theta^{-1}d^*_{ij}, \tag{3.2.8h}$$
$$\acute{h}_{ij} \equiv \theta^2 h_{ij}, \tag{3.2.8i}$$

are also a well-defined solution of the CCEs. Note that contractions on the right-hand-side are made with respect to the metric $\boldsymbol{h}$ and its inverse. More specifically, transforming

$$(\dagger): \qquad (\Omega, \sigma_i, \sigma, s, L_i, L_{ij}, K_{ij}, d^*_{ij}, d_{ij}, h_{ij}) \mapsto (\acute{\Omega}, \acute{\sigma}_i, \acute{\sigma}, \acute{s}, \acute{L}_i, \acute{L}_{ij}, \acute{K}_{ij}, \acute{d}^*_{ij}, \acute{d}_{ij}, \acute{h}_{ij}),$$

then the new zero quantities

$$(\acute{P}_{ij}, \acute{Z}_i, \acute{W}_i, \acute{X}_{ij}, \acute{X}_{ijk}, \acute{Y}_{ijk}, \acute{\Lambda}^*_i, \acute{\Lambda}_i, \acute{U}_{ij}, \acute{A}),$$

are given in terms of the original zero quantities by

$$\acute{P}_{ij} = \theta P_{ij}, \qquad \acute{Z}_i = Z_i, \qquad \acute{Y}_{ijk} = \theta Y_{ijk},$$

$$\acute{\Lambda}_i = \theta^{-3}\Lambda_i, \qquad \acute{\Lambda}^*_i = \theta^{-3}\Lambda^*_i, \qquad \acute{U}_{ij} = U_{ij}, \qquad \acute{A} = A,$$

and

$$\begin{aligned}
\acute{W}_i &= \theta^{-1}W_i - \theta^{-1}\phi Z_i + \theta^{-2}Z_i{}^jD_j\theta,\\
\acute{X}_{ij} &= \theta^{-1}X_{ij} + \theta^{-2}(D^k\theta)Y_{ijk},\\
\acute{X}_{ijk} &= X_{ijk} + \phi Y_{ijk} + \theta^{-1}U_{jk}D_i\theta - \theta^{-1}U_{ik}D_j\theta - \theta^{-1}U_{jl}h_{ik}D^l\theta + \theta^{-1}U_{il}h_{jk}D^l\theta.
\end{aligned}$$

The expressions are homogeneous in the original zero quantities, and hence the new zero quantities vanish if the original ones vanish —that is to say that the fields (3.2.8a)–(3.2.8i) also comprise a solution to the CCEs, as claimed above. We refer to the above as *quasi conformal covariance* to reflect the fact that the transformation (†) involves not only θ and ϕ but also their derivatives. Conformal covariance of the CCEs can be seen as being inherited by conformal covariance of the CFEs —performing a rescaling $\Xi \to \Theta\Xi$ for some $\Theta > 0$ (i.e. rescaling the unphysical spacetime metric to $\tilde{\boldsymbol{g}} \equiv \Theta^2\boldsymbol{g}$), and performing the implied transformation of the variables s, L_{ab}, d_{abcd}, induces the transformation (†) on the constraint variables, with $\theta > 0$ and ϕ given by

$$\theta \equiv \Theta|_{\mathcal{S}}, \qquad \phi \equiv \boldsymbol{n}(\Theta)|_{\mathcal{S}}.$$

The field θ therefore represents an *intrinsic conformal freedom* in the CCEs, while ϕ represents an *extrinsic conformal freedom*. For later use, it will be useful to note the transformation for the mean extrinsic curvature:

$$\acute{K} = \theta^{-1}(K + 3\phi), \tag{3.2.9}$$

which may be recovered by tracing (3.2.8f). Given a solution to the CCEs, one can use the transformation (†) to recover the corresponding *physical* solution—i.e. the resulting solution of the Einstein constraints— by setting $\theta = \Omega^{-1}$ and $\phi = -\Omega^{-1}\sigma$. Immediately we see that $\tilde{\Omega} = 1,\ \tilde{\sigma} = 0$. For the Friedrich scalar, we see using $A = 0$ that

$$\tilde{s} = \Omega s + \tfrac{1}{2}\sigma^2 - \tfrac{1}{2}D_i\Omega D^i\Omega = \tfrac{\lambda}{6} \tag{3.2.10}$$

For the tangential components of the Schouten tensor, we find

$$\begin{aligned}\tilde{L}_{ij} &= L_{ij} - \Omega^{-1}K_{ij}\sigma + \tfrac{1}{2}\Omega^{-2}h_{ij}\sigma^2 + \Omega^{-1}D_iD_j\Omega - \tfrac{1}{2}\Omega^{-2}h_{ij}D_k\Omega D^k\Omega,\\ &= \tfrac{1}{2}\Omega^{-2}(2\Omega s - D_k\Omega D^k\Omega + \sigma^2)h_{ij}\\ &= \tfrac{1}{6}\lambda\tilde{h}_{ij}\end{aligned} \tag{3.2.11}$$

where we have used $Z_{ij} = Q_i = 0$ in the second equality and $A = 0$, along with $\tilde{h}_{ij} = \Omega^{-2}h_{ij}$, in the third. On the other hand, the normal-tangential component of the Schouten tensor vanishes

$$\tilde{L}_i = D_i\sigma + \Omega L_i - K_{ij}D^j\Omega = 0, \tag{3.2.12}$$

the second equality following from $Z_i = 0$. This is not surprising, since the CFE equation from which $Z_i = 0$ is derived is basically the transformation formula for L_{ab}. Finally, we have from (3.2.8f)–(3.2.8h) that

$$\tilde{K}_{ij} = \Omega^{-1}K_{ij} - \Omega^{-2}\sigma h_{ij}, \tag{3.2.13}$$

$$\tilde{d}_{ij} = \Omega d_{ij}, \tag{3.2.14}$$

$$\tilde{d}^*_{ij} = \Omega d^*_{ij}. \tag{3.2.15}$$

Substituting, we find that, trivially

$$\tilde{P}_{ij} \equiv 0,\ \tilde{Z}_i \equiv 0,\ \tilde{W}_i \equiv 0,\ \tilde{X}_{ij} \equiv 0,\ \tilde{X}_{ijk} \equiv 0,\ \tilde{A} \equiv 0,$$

and the remaining zero quantities reduce to

$$\tilde{Y}_{ijk} \equiv \tilde{D}_i\tilde{K}_{jk} - \tilde{D}_j\tilde{K}_{ik} - \tilde{\epsilon}^l{}_{ij}\tilde{d}^*_{kl}, \tag{3.2.16a}$$

$$\tilde{\Lambda}_i \equiv \tilde{D}_j\tilde{d}_i{}^j - \tilde{\epsilon}_{ikl}\tilde{K}^{jk}\tilde{d}^{*l}_j, \tag{3.2.16b}$$

$$\tilde{\Lambda}^*_l \equiv \tilde{D}^i\tilde{d}^*_{il} + \tilde{\epsilon}_{lkj}\tilde{K}_i{}^k\tilde{d}^{ij}, \tag{3.2.16c}$$

$$\tilde{U}_{ij} \equiv \tilde{l}_{ij} - \tfrac{\lambda}{6}\tilde{h}_{ij} - \tilde{d}_{ij} - \tilde{K}_i{}^k\tilde{K}_{jk} + \tfrac{1}{4}\tilde{K}_{kl}\tilde{K}^{kl}\tilde{h}_{ij} - \tfrac{1}{4}\tilde{K}^2\tilde{h}_{ij} + \tilde{K}\tilde{K}_{ij}, \tag{3.2.16d}$$

with all index lowering and raising with respect to the physical metric, $\tilde{h}_{ij}$, and its inverse. A slightly modified but equivalent system, referred to as the *Extended Constraint Equations*, will be studied in more detail later —see Section 3.3.2. The trace parts $\tilde{h}^{ik}\tilde{Y}_{ijk} = 0$ and $\tilde{h}^{ij}\tilde{U}_{ij} = 0$ are precisely the Einstein constraint equations for $\tilde{h}_{ij}, \tilde{K}_{ij}$. It is clear from the above discussion that any two solutions of the CCEs that are related by (†) give rise to the same physical solution. Hence, there is a bijection between the set of solutions of the Einstein constraints and the set of equivalence classes (with equivalence relation (†)) of solutions of the CCEs.

3.2.3 The integrability conditions

While the CCEs are a highly-coupled system of far greater complexity than the Einstein constraints, the equations are not entirely independent —this is a result of the fact that they are derived from the CFEs which are constructed, in a sense, by appending to the Einstein field equation various integrability conditions (c.f. the construction of the equation $Z_a = 0$). More precisely, the CCE zero quantities can be seen to satisfy the following integrability conditions

$$\begin{aligned}
\epsilon_{lik} D^k P_j{}^i &= \epsilon_{jli} W^i - \tfrac{1}{2}\Omega\epsilon_{lik}\left(X^{ik}{}_j - 2K_j{}^k Z^i - \tfrac{1}{2}\sigma Y^{ik}{}_j\right) \\
&\qquad + \epsilon_{jlk}\left(D^i\Omega\right) U_i{}^k + \epsilon_{lik}\left(D^i\Omega\right) U_j{}^k, && \text{(3.2.17a)} \\
D_i Z_j - D_j Z_i &= \Omega X_{ij} + K_{ik} P_j{}^k - K_{jk} P_i{}^k - \left(D^k\Omega\right) Y_{ijk}, && \text{(3.2.17b)} \\
D_i W_j - D_j W_i &= L_i Z_j - L_j Z_i - L_{ik} P_j{}^k + L_{jk} P_i{}^k - \sigma X_{ij} + \left(D^k\Omega\right) X_{ijk}, && \text{(3.2.17c)} \\
\epsilon_{ijk} D^k X^{ij} &= \epsilon_{ijl} K_k{}^l X^{ijk} - 2d^*_{ij} P^{ij} - \epsilon_{ijl} L_k{}^l Y^{ijk} - 2\left(D_i\Omega\right)\Lambda^{*i}, && \text{(3.2.17d)} \\
\epsilon_{ijk} D^k X^{ij}{}_l &= -2\epsilon_{ljk} d_i{}^k P^{ij} + \epsilon_{ijk} K_l{}^k X^{ij} - 2d^*_{li} Z^i - \epsilon_{ijk} L^i Y^{jk}{}_l \\
&\qquad + 2\epsilon_{ljk} L_i{}^k U^{ij} - 2\sigma\Lambda^*_l + 2\epsilon_{lij}(D^j\Omega)\Lambda^i, && \text{(3.2.17e)} \\
\epsilon_{ijk} D^k Y^{ij}{}_l &= -2\Omega\Lambda^*_l + \epsilon_{lij} X^{ij} - 2\epsilon_{ljk} K^{ij} U_i{}^k, && \text{(3.2.17f)} \\
D_i U_j{}^i - D_j U^i{}_i &= X_j{}^i{}_i - \Omega\Lambda_j + K_{ik} Y_j{}^{ik} - K_{jk} Y^{ik}{}_i - K Y_j{}^i{}_i, && \text{(3.2.17g)} \\
D_i A &= -6\Omega W_i - 6\sigma Z_i + 6(D^j\Omega) P_{ij}. && \text{(3.2.17h)}
\end{aligned}$$

Relations (3.2.17a)–(3.2.17c) and (3.2.17f)–(3.2.17h) were noted already in [33] but (3.2.17d) and (3.2.17e) appear to be new. We therefore present only the derivation of the integrability relations (3.2.17d) and (3.2.17e).

We first note the commutation relations

$$\begin{aligned}
\epsilon_{ljk} D^k D^j L^i &= \epsilon_{ljk} l^{ik} L^j + \epsilon^i{}_{lk} l_j{}^k L^j, && \text{(3.2.18a)} \\
\epsilon_{mij} D^j D^i L_{kl} &= \epsilon_{lmj} l^{ij} L_{ki} - \epsilon_{mij} l_l{}^i L_k{}^j + \epsilon_{kmj} l^{ij} L_{li} - \epsilon_{mij} l_k{}^i L_l{}^j, && \text{(3.2.18b)}
\end{aligned}$$

the following substitutions of the zero quantities

$$\begin{aligned}
\epsilon_{mij} D^j L^i &= -\tfrac{1}{2} X^{ij}\epsilon_{mij} - \epsilon_{mjk} K^{ij} L_i{}^k - \tfrac{1}{2}\epsilon_{mjk} d_i{}^{jk} D^i\Omega, && \text{(3.2.19a)} \\
\epsilon_{mij} D^j L_k{}^i &= -\tfrac{1}{2} X^{ij}{}_k \epsilon_{mij} + \epsilon_{mij} K_k{}^j L^i - d^*_{km}\sigma + \epsilon d_i{}^j \epsilon_{kmj} D^i\Omega - d_k{}^j \epsilon_{mij} D^i\Omega, && \text{(3.2.19b)} \\
\epsilon_{lij} D^j K_k{}^i &= -\tfrac{1}{2} Y^{ij}{}_k \epsilon_{lij} - \Omega d^*_{kl} + \epsilon_{kli} L^i, && \text{(3.2.19c)}
\end{aligned}$$

which follow immediately from the definitions of X_{ij}, X_{ijk}, Y_{ijk}, and the identities

$$\begin{aligned}
\epsilon_{ijk} D^k d_l{}^j - \epsilon_{ljk} D^k d_i{}^j &= -\epsilon_{ilk} D^j d_j{}^k && \text{(3.2.20a)} \\
\epsilon_{ljk} K K^{ij} L_i{}^k - \epsilon_{lkm} K_i{}^k K^{ij} L_j{}^m - \epsilon_{ikm} K^{jk} K_l{}^i L_j{}^m &= 0. && \text{(3.2.20b)}
\end{aligned}$$

To verify (3.2.20a) and (3.2.20b), contract them with $\epsilon^{il}{}_m$ and $\epsilon^l{}_{mn}$, respectively, and use formula

(2.1.8b) for the products of volume forms. The derivation of (3.2.17d) is then as follows

$$\begin{aligned}
\epsilon_{ijk}D^kX^{ij} &= -2D^i\Omega D_j d_i^{*j} - 2d^*_{ij}D^jD^i\Omega - 2\epsilon_{ijk}D^kD^jL^i + 2\epsilon_{jkl}L^{ij}D^lK_i{}^k - 2\epsilon_{jkl}K^{ij}D^lL_i{}^k \\
&= -2P^{ij}d^*_{ij} + 2\Omega L^{ij}d^*_{ij} - 2\sigma K^{ij} - 2D^i\Omega D_j d_i^{*j} \\
&\qquad - 2\epsilon_{ijk}D^kD^jL^i + 2\epsilon_{jkl}L^{ij}D^lK_i{}^k - 2\epsilon_{jkl}K^{ij}D^lL_i{}^k) \\
&= -2P^{ij}d^*_{ij} + 2\Omega d^*_{ij}L^{ij} - 2\epsilon K^{ij}d^*_{ij}\sigma - 2D^i\Omega D_j d_i^{*j} + 2\epsilon_{jkl}L^{ij}D^lK_i{}^k - 2\epsilon_{jkl}K^{ij}D^lL_i{}^k \\
&= -2P^{ij}d^*_{ij} - Y^{ijk}\epsilon_{ijl}L_k{}^l + 2\epsilon K^{ij}d^*_{ij}\sigma - 2D^i\Omega D_j d_i^{*j} - 2\epsilon_{jkl}K^{ij}D^lL_i{}^k \\
&= X^{ijk}\epsilon_{ijl}K_k{}^l - 2P^{ij}d^*_{ij} - Y^{ijk}\epsilon_{ijl}L_k{}^l - 2d^{jk}\epsilon_{ikl}K_j{}^lD^i\Omega - 2D^i\Omega D_j d_i^{*j} \\
&= X^{ijk}\epsilon_{ijl}K_k{}^l - 2P^{ij}d^*_{ij} - Y^{ijk}\epsilon_{ijl}L_k{}^l - 2\Lambda^*_i D^i\Omega.
\end{aligned}$$

The first equality is by definition of X_{ij}, the second is by substituting for $D_iD_j\Omega$ using the definition of P_{ij}, the third, fourth and fifth use (3.2.18a), (3.2.19c) and (3.2.19b), and the final equality follows by substituting for the $D^id^*_{ij}$ term using the definition of Λ^*_i.

On the other hand, (3.2.17d) can be derived as follows

$$\begin{aligned}
\epsilon_{ijk}D^kX^{ij}{}_l &= -\epsilon_{ijk}d_l{}^{jk}D^i\sigma + 2\epsilon_{lkj}D^i\Omega D^jd_i{}^k - 2\epsilon_{ijk}K_l{}^kD^jL^i + 2d_l{}^k\epsilon_{ijk}D^jD^i\Omega \\
&\qquad - 2d_i{}^k\epsilon_{jkl}D^jD^i\Omega - 2\epsilon_{ijk}D^i\Omega D^kd_l{}^j + 2\epsilon_{ijk}L^iD^kK_l{}^j - \epsilon_{ijk}\sigma D^kd_l{}^{ij} - \epsilon_{ijk}D^kD^jL_l{}^i \\
&= -2P^{ij}d_i{}^k\epsilon_{ljk} + X^{ij}\epsilon_{ijk}K_l{}^k - 2Z^id^*_{li} - Y^{jk}{}_l\epsilon_{ijk}L^i - 2\Omega d^{ij}\epsilon_{ljk}L_i{}^k \\
&\qquad + 2\epsilon_{ikm}K^{jk}K_l{}^iL_j{}^m + 2d^{ij}\epsilon_{ljk}K_i{}^k\sigma - 2\sigma D_id_l^{*i} + 2K_l{}^jd^*_{ij}D^i\Omega \\
&\qquad - 2K_i{}^jd^*_{lj}D^i\Omega + 2\epsilon_{ljk}D^i\Omega D^kd_i{}^j - 2\epsilon_{ijk}D^i\Omega D^kd_l{}^j - 2\epsilon_{ijk}D^kD^jL_l{}^i \\
&= -2P^{ij}d_i{}^k\epsilon_{ljk} + X^{ij}\epsilon_{ijk}K_l{}^k - 2Z^id^*_{li} - Y^{jk}{}_l\epsilon_{ijk}L^i + 2\epsilon_{ljk}U^{ij}L_i{}^k \\
&\qquad + 2\epsilon_{lkm}K_i{}^kK^{ij}L_j{}^m + 2\epsilon_{ikm}K^{jk}K_l{}^iL_j{}^m - 2\Lambda^*_l\sigma - 2\epsilon_{ljk}K^{ij}L_i{}^kK \\
&\qquad + 2K_l{}^jd^*_{ij}D^i\Omega - 2K_i{}^jd^*_{lj}D^i\Omega + 2\epsilon_{ljk}D^i\Omega D^kd_i{}^j - 2\epsilon_{ijk}D^i\Omega D^kd_l{}^j \\
&= -2P^{ij}d_i{}^k\epsilon_{ljk} + X^{ij}\epsilon_{ijk}K_l{}^k - 2Z^id^*_{li} - Y^{jk}{}_l\epsilon_{ijk}L^i + 2\epsilon_{ljk}U^{ij}L_i{}^k \\
&\qquad + 2\epsilon_{lkm}K_i{}^kK^{ij}L_j{}^m + 2\epsilon_{ikm}K^{jk}K_l{}^iL_j{}^m - 2\Lambda^*_l\sigma - 2\epsilon_{ljk}K^{ij}L_i{}^kK \\
&\qquad + 2K_l{}^jd^*_{ij}D^i\Omega - 2K_i{}^jd^*_{lj}D^i\Omega + 2\epsilon_{ljk}D^i\Omega D^kd_i{}^j - 2\epsilon_{ijk}D^i\Omega D^kd_l{}^j \\
&= -2P^{ij}d_i{}^k\epsilon_{ljk} + X^{ij}\epsilon_{ijk}K_l{}^k - 2Z^id^*_{li} - Y^{jk}{}_l\epsilon_{ijk}L^i + 2\epsilon_{ljk}U^{ij}L_i{}^k \\
&\qquad + 2\epsilon_{lkm}K_i{}^kK^{ij}L_j{}^m + 2\epsilon_{ikm}K^{jk}K_l{}^iL_j{}^m - 2\Lambda^*_l\sigma - 2\epsilon_{ljk}K^{ij}L_i{}^kK + 2\Lambda^i\epsilon_{lij}D^j\Omega \\
&= -2P^{ij}d_i{}^k\epsilon_{ljk} + X^{ij}\epsilon_{ijk}K_l{}^k - 2Z^id^*_{li} - Y^{jk}{}_l\epsilon_{ijk}L^i + 2\epsilon_{ljk}U^{ij}L_i{}^k - 2\Lambda^*_l\sigma + 2\Lambda^i\epsilon_{lij}D^j\Omega.
\end{aligned}$$

The first equality is by definition of X_{ijk}, the second follows by substituting for $D_i\sigma$ and $D_iD_j\Omega$ using the definitions of Z_i and P_{ij}, the third follows from (3.2.18b), substituting for $D^jd^*_{ij}$ and l_{ij} using Λ^*_i and U_{ij}, the fourth follows from substituting for D^jd_{ij} using Λ_i and using (3.2.19a), the fifth follows from (3.2.20a) and substituting D^jd_{ij} using Λ_i, and finally the sixth follows from (3.2.20b).

3.3 Simplifications of the Conformal Constraint Equations

Here we present some noteworthy simplifications/reductions of the CCEs. The greatest simplification, described in [33], is obtained when the hypersurface comprises part of the conformal boundary, $\mathcal{S} \subseteq \mathscr{I}$. Interesting examples include the timelike conformal boundary of anti-de Sitter–like spacetimes and the spacelike future null infinity of de Sitter–like spacetimes. Prescription of initial data for the latter is part of what is sometimes called the *aymptotic initial value problem.* Since we require $\Xi = 0$ on $\mathscr{I}$, it follows that for $\mathcal{S} \subseteq \mathscr{I}$ we require $\Omega = 0,\ D_i\Omega = 0$ and the CCEs simplify enough to permit an explicit solution, once the relation between s and σ has been fixed —see [34] for more details.

Although it will not be explored further in this thesis, we will first describe what we call the "umbilical" reduction of the CCEs. There are three reasons for including this section: firstly, a systematic derivation seems to be absent from the literature; secondly, the *semi-global existence* result of H. Friedrich (see [35]) for perturbations of hyperboloidal data is one of the chief motivations for studying the CCEs; thirdly, the reduction involves a gauge-fixing procedure for the extrinsic conformal freedom —see Section 3.2.2— which will be generalised in order to perform an elliptic reduction of the full CCEs in Chapter 7.

3.3.1 Umbilical initial data

An *umbilical initial data set* is one for which the extrinsic curvature is pure-trace:

$$\tilde{K}_{ij} = \tfrac{1}{3}\tilde{K}\tilde{h}_{ij}.$$

Note that the scalar $\tilde{K}$ is necessarily constant as a consequence of the momentum constraint (1.2.1b). Umbilical initial data sets generalise the hyperboloids of Minkowski space,

$$\mathcal{H}_k = \{p \in \mathbb{R}^4 \mid t(p)^2 - r(p)^2 = k\}, \qquad k > 0,$$

and are themselves the simplest examples of the more general class of *hyperboloidal* initial data.[2] In addition to the role they play in Friedrich's semi-global existence result, hyerboloidal data sets (or more precisely, hyperboloidal foliations) have also been used in numerical investigations since they are adapted, in a certain sense, to the extraction of gravitational radiation —see [47–49].

Definition 8. A triple $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ satisfying the vacuum Einstein constraint equations with $\lambda = 0$ is said to be *hyperboloidal* if

(i) there exists a compactification $\mathcal{S}$, diffeomorphic to the closed unit ball in $\mathbb{R}^3$, whereby $\tilde{\mathcal{S}}$ is identified with $\mathcal{S} \setminus \partial\mathcal{S}$;

(ii) there exist functions Ω, σ on $\mathcal{S}$ such that $\Omega|_{\tilde{\mathcal{S}}} > 0$ and $\Omega = 0,\ \sigma \neq 0$ on $\partial\mathcal{S}$;

(iii) the conformal fields

$$\boldsymbol{h} = \Omega^{-2}\tilde{\boldsymbol{h}}, \qquad \boldsymbol{K} = \Omega(\tilde{\boldsymbol{K}} + \sigma\tilde{\boldsymbol{h}}).$$

extend smoothly to $\mathcal{S}$, and

$$\sigma^2 \equiv \mathbf{h}^\sharp(\mathbf{d}\Omega, \mathbf{d}\Omega)$$

on $\partial\mathcal{S}$.

Note that the last condition is simply that $A = 0$ on $\partial\mathcal{S}$. Note also that, for a given umbilical initial data set, each of related unphysical solutions is also umbilical, which may be seen using the transformation law for the extrinsic curvature, (2.2.10):

$$K_{ij} = \Omega\tilde{K}_{ij} + \Omega\sigma\tilde{h}_{ij} = \tfrac{1}{3}\Omega(\tilde{K} + 3\sigma)\tilde{h}_{ij} = \tfrac{1}{3}\Omega^{-1}(\tilde{K} + 3\sigma)h_{ij}.$$

[2] Not to be confused with the "hyperbolic" initial data sets which are considered in Chapter 4.

In other words, we find that $K_{ij} = \frac{1}{3}Kh_{ij}$, with

$$K \equiv \mathrm{tr}_{\boldsymbol{h}}\boldsymbol{K} = \Omega^{-1}(\tilde{K} + 3\sigma). \tag{3.3.1}$$

This is precisely the transformation formula for the mean extrinsic curvature, (3.2.9).

Reduction of the CCEs for umbilical initial data sets

First note that substituting $K_{ij} = \frac{1}{3}Kh_{ij}$ into $Y_{ijk} = 0$, we immediately obtain

$$d^*_{ij} = 0, \qquad L_i = -\tfrac{1}{3}D_iK.$$

Then, substituting into Z_i we find that $Z_i = 0$ is equivalent to

$$D_i(\Omega K - 3\sigma) = 0,$$

implying that $\Omega K - 3\sigma$ is constant on $\mathcal{S}$; clearly $\Omega K - 3\sigma$ can be identified with $\tilde{K}$ using (3.3.1), and so we recover the fact that $\tilde{K} = \Omega K - 3\sigma$ is constant (as remarked earlier). Taking $\tilde{K} = \Omega K - 3\sigma$ to be constant, then, the zero quantities Z_i, Y_{ijk}, X_{ij} trivialise and the remaining non-trivial CCEs take the following form

$$P_{ij} \equiv D_iD_j\Omega + \Omega L_{ij} - \tfrac{1}{9}K(\Omega K - \tilde{K})h_{ij} - sh_{ij}, \tag{3.3.2a}$$
$$W_i = D_is + D^j\Omega L_{ij} + \tfrac{1}{9}(\Omega K - \tilde{K})D_iK, \tag{3.3.2b}$$
$$X_{ijk} \equiv D_iL_{jk} - D_jL_{ik} - \tfrac{1}{9}h_{kj}KD_iK + \tfrac{1}{9}h_{ki}KD_jK + 2d_{k[i}D_{j]}\Omega - 2d_{l[i}h_{j]k}D^l\Omega, \tag{3.3.2c}$$
$$\Lambda_i \equiv D^jd_{ij}, \tag{3.3.2d}$$
$$U_{ij} \equiv l_{ij} - L_{ij} - \Omega d_{ij} + \tfrac{1}{18}K^2h_{ij}, \tag{3.3.2e}$$
$$A \equiv \lambda - 6\Omega s + 3D_i\Omega D^i\Omega - \tfrac{1}{3}(\Omega K - \tilde{K})^2. \tag{3.3.2f}$$

Note that there is no equation for K, and hence K needs to be prescribed. The scalar K can be thought of as a *conformal gauge-source function* —i.e. it encodes the extrinsic conformal freedom. Given K, the remaining fields are fixed as follows

$$K_{ij} = \tfrac{1}{3}Kh_{ij}, \qquad d^*_{ij} = 0, \qquad L_i = -\tfrac{1}{3}D_iK, \qquad \sigma = \tfrac{1}{3}(\Omega K - \tilde{K}). \tag{3.3.3}$$

It is clear that since the CCEs are preserved under (†) and since the property of being umbilical is invariant under (†), all conformally-related unphysical solutions (corresponding to a given *physical* umbilical initial data set) take the form (3.3.3). We can check this directly using (†):

$$\begin{aligned}
\acute{L}_i &= \theta^{-1}L_i + \theta^{-2}\phi D_i\theta - \theta^{-1}D_i\phi + \theta^{-2}K_{ij}D^j\theta \\
&= -\tfrac{1}{3}\theta^{-1}D_iK + \theta^{-2}\phi D_i\theta - \theta^{-1}D_i\phi + \tfrac{1}{3}\theta^{-2}KD_i\theta \\
&= -\tfrac{1}{3}D_i(K + 3\phi) + \tfrac{1}{3}\theta^{-2}(K + 3\phi)D_i\theta \\
&= -\tfrac{1}{3}D_i(\theta^{-1}(K + 3\phi)) \\
&= -\tfrac{1}{3}\acute{D}_i\acute{K},
\end{aligned}$$

while

$$\begin{aligned}\acute{\sigma} = \sigma + \Omega\phi = \tfrac{1}{3}(\Omega K - \tilde{K}) + \Omega\phi &= \tfrac{1}{3}(\Omega(K + 3\phi) - \tilde{K}) \\ &= \tfrac{1}{3}(\acute{\Omega}\theta^{-1}(K + 3\phi) - \tilde{K}) \\ &= \tfrac{1}{3}(\acute{\Omega}\acute{K} - \tilde{K}).\end{aligned}$$

Taking advantage of the conformal covariance, the umbilical CCEs can be further simplified by a convenient choice of extrinsic conformal gauge. Indeed, given a solution of (3.3.2a)–(3.3.2f),

$$(\Omega, s, L_{ij}, d_{ij}, h_{ij}),$$

we can perform a subsequent conformal rescaling with

$$\theta = 1, \qquad \phi = -\tfrac{1}{3}K.$$

Substituting into the conformal transformation formula (3.2.9) for K, we find that

$$\acute{K} = \theta\acute{K} = K + 3\phi = 0,$$

so that the resulting conformal representation is *maximal*. Using (3.3.3), we then obtain (immediately dropping the tildes) the following *gauge-reduced* system

$$P_{ij} \equiv D_i D_j \Omega + \Omega L_{ij} - s h_{ij}, \tag{3.3.4a}$$

$$W_i = D_i s + D^j \Omega L_{ij}, \tag{3.3.4b}$$

$$X_{ijk} \equiv D_i L_{jk} - D_j L_{ik} + 2d_{k[i} D_{j]}\Omega - 2d_{l[i} h_{j]k} D^l \Omega, \tag{3.3.4c}$$

$$\Lambda_i \equiv D^j d_{ij}, \tag{3.3.4d}$$

$$U_{ij} \equiv l_{ij} - L_{ij} - \Omega d_{ij}, \tag{3.3.4e}$$

$$A \equiv \lambda - 6\Omega s + 3\sigma_i \sigma^i - \tfrac{1}{3}\tilde{K}^2, \tag{3.3.4f}$$

with the remaining fields being given by

$$K_{ij} = 0, \qquad d^*_{ij} = 0, \qquad L_i = 0, \qquad \sigma = -\tfrac{1}{3}\tilde{K}.$$

Note that the influence of $\tilde{K}$, which is now a prescribed constant, is only felt through the algebraic constraint. As a special case, one of course recovers the time symmetric case by setting $\tilde{K} = 0$. This is one example of conformal gauge fixing, which will be generalised to the full CCEs in Section 7.1.1. Note that the intrinsic conformal freedom has not been used, here. Equations (3.3.4a)–(3.3.4f) can be regarded as a conformally-regular version of the conformal Hamiltonian constraint, just as the CFEs (with which they bear a formal resemblance) are a conformally-regular version of the Einstein field equations.

Since the zero quantities Z_i, X_{ij}, Y_{ijk} and Λ^*_i vanish trivially, the integrability relations (3.2.17a)–

(3.2.17h) reduce to

$$\epsilon_{lik}D^kP_j{}^i = \epsilon_{jli}W^i - \tfrac{1}{2}\Omega\epsilon_{lik}X^{ik}{}_j + \epsilon_{jlk}D^i\Omega U_i{}^k + \epsilon_{lik}D^i\Omega U_j{}^k, \tag{3.3.5a}$$
$$D_iW_j - D_jW_i = -L_{ik}P_j{}^k + L_{jk}P_i{}^k + D^k\Omega X_{ijk}, \tag{3.3.5b}$$
$$\epsilon_{ijk}D^kX^{ij}{}_l = -2\epsilon_{ljk}d_i{}^kP^{ij} + 2\epsilon_{ljk}L_i{}^kU^{ij} + 2\epsilon_{lij}D^j\Omega\Lambda^i, \tag{3.3.5c}$$
$$D_iU_j{}^i - D_jU^i{}_i = X_j{}^i{}_i - \Omega\Lambda_j, \tag{3.3.5d}$$
$$D_iA = -6\Omega W_i + 6D^j\Omega P_{ij}. \tag{3.3.5e}$$

One method of attempting to solve the umbilical constraint equations is as follows: one first prescribes a metric, h_{ij}, and then solves the conformal momentum constraint for Ω. To do this, take $\Omega = \rho\vartheta^{-2}$ where ρ is a given *boundary defining function* —i.e. a function satisfying

$$\rho|_{\partial\mathcal{S}} = 0, \qquad d\rho|_{\partial\mathcal{S}} \neq 0$$

— whereupon the conformal momentum constraint reduces to

$$\rho^2\Delta_{\boldsymbol{h}}\vartheta - \rho D_i\rho D^i\vartheta + \left(\tfrac{3}{2}D_iD^i\rho - \tfrac{1}{8}r[\boldsymbol{h}]\vartheta\rho^2 - \tfrac{1}{2}\rho\vartheta^2\Delta_{\boldsymbol{h}}\rho\right)\vartheta = \tfrac{1}{8}\tilde{K}^2\vartheta^{-5}, \tag{3.3.6}$$

thought of now as an equation for ϑ. One then solves $P_{ij} = 0$ and $U_{ij} = 0$ algebraically, to obtain

$$s = \tfrac{1}{3}\Delta\Omega + \tfrac{1}{12}\Omega r, \tag{3.3.7a}$$
$$L_{ij} = -\Omega^{-1}D_{\{i}D_{j\}}\Omega + \tfrac{1}{12}\Omega r h_{ij}, \tag{3.3.7b}$$
$$d_{ij} = \Omega^{-2}D_{\{i}D_{j\}}\Omega + \Omega^{-1}r_{\{ij\}}, \tag{3.3.7c}$$

with $\tilde{K}$ chosen so as to satisfy $A = 0$. It can easily be verified that the remaining equations, $W_i = \Lambda_i = 0$, $X_{ijk} = 0$, are satisfied automatically by virtue of the integrability conditions. Of course, it is not clear a priori whether ϑ, L_{ij} and d_{ij} extend smoothly to the conformal boundary. The following theorem of [50] addresses this issue.

Theorem. (Andersson, Chruściel, Friedrich) Let $(\mathcal{S}, \boldsymbol{h})$ be a 3−dimensional orientable, compact, smooth Riemannian manifold with boundary $\partial\mathcal{S}$. Then there exists a unique solution, ϑ, of (3.3.6) and the following are equivalent

(i) The fields ϑ along with s, L_{ij}, d_{ij}, as determined by (3.3.7a)–(3.3.7c), on $\mathcal{S}\setminus\partial\mathcal{S}$ extend smoothly to $\mathcal{S}$,

(ii) The electric part of the Weyl tensor, $S_{ij} \equiv \Omega d_{ij}$, goes to zero at $\partial\mathcal{S}$,

(iii) The conformal class of $\boldsymbol{h}$ is such that the extrinsic curvature of $\partial\mathcal{S} \hookrightarrow \mathcal{S}$ is pure-trace (with respect to the induced metric).

3.3.2 The Extended Constraint Equations

In Section 3.2.2, we saw that by applying the conformal transformation (†) with $\theta = \Omega^{-1}$, $\phi = -\Omega^{-1}\sigma$, the CCEs reduce to the equations

$$\tilde{Y}_{ijk} = 0, \qquad \tilde{\Lambda}^*_i = 0, \qquad \tilde{\Lambda}_i = 0, \qquad \tilde{U}_{ij} = 0. \tag{3.3.8}$$

These equations, which may of course be derived more directly simply by setting $\Omega = 1,\ \sigma = 0$ in the CCEs, are to be read as a mixed-order system of PDEs for the physical 3-metric and extrinsic curvature, $\tilde{h}_{ij},\ \tilde{K}_{ij}$, and the fields $\tilde{d}_{ij},\ \tilde{d}^*_{ij}$. Setting $\Omega = 1$ (and dropping the tildes, for convenience) we see that $d^a{}_{bcd} = C^a{}_{bcd}$ on $\mathcal{S}$, and so d_{ij} and d^*_{ij} are equal to the electric and magnetic parts of the Weyl curvature, which we now denote by S_{ij} and $\bar{S}_{ij}$, in keeping with the notation of [28, 29]. For reference, the equations now read

$$J_{ijk} \equiv D_i K_{jk} - D_j K_{ik} - \epsilon^l{}_{ij} \bar{S}_{kl} = 0, \tag{3.3.9a}$$
$$\Lambda_i \equiv D_j S_i{}^j - \epsilon_{ikl} K^{jk} \bar{S}_j{}^l = 0, \tag{3.3.9b}$$
$$\Lambda^*_l \equiv D^i \bar{S}_{il} - \epsilon_{ljk} K_i{}^k S^{ij} = 0, \tag{3.3.9c}$$
$$U_{ij} \equiv l_{ij} - \tfrac{\lambda}{6} h_{ij} - S_{ij} - K_i{}^k K_{jk} + \tfrac{1}{4}(K_{kl}K^{kl} - K^2) h_{ij} + K K_{ij} = 0. \tag{3.3.9d}$$

Note that we are now using "J_{ijk}" to label the zero quantity, rather than "Y_{ijk}", in order to avoid confusion with the Cotton tensor. Equations (3.3.9a) and (3.3.9d) are precisely the Codazzi–Mainardi and Gauss–Codazzi equations for the embedding $\mathcal{S} \hookrightarrow \mathcal{M}$ —see Section 2.1.2. The remaining two electromagnetic constraints (3.3.9b) and (3.3.9c), can be seen as the spatial projections of the second Bianchi identity of the ambient spacetime manifold, which (assuming that the vacuum Einstein field equations hold) reads

$$\nabla_a C^a{}_{bcd} = 0.$$

To see this, use the electromagnetic decomposition of the Weyl tensor and the projector formalism from Section 2.1.2. The integrability conditions (3.2.17a)–(7.1.2h) reduce to

$$\epsilon_{ijk} D^k J^{ij}{}_l = -2\Lambda^*_l - 2\epsilon_{ljk} K^{ij} U_i{}^k, \tag{3.3.10a}$$
$$D^i U_{ij} - D_j U_i{}^i = -\Lambda_j + J_j{}^{ik} K_{ik} - J^{ik}{}_i K_{jk} - K J_j{}^i{}_i \tag{3.3.10b}$$

—the remaining integrability conditions trivialise as a result of the trivialisation of their constituent zero quantities. In particular, if $J_{ijk} = 0$ and $U_{ij} = 0$, then it automatically follows that $\Lambda_i = \Lambda^*_i = 0$. Rather than working with equations (3.3.9a)–(3.3.9d), a slightly modified version of the equations will turn out to be more convenient. First, we replace $U_{ij} = 0$ with the equivalent equation $V_{ij} = 0$, where

$$V_{ij} \equiv U_{ij} + (\mathrm{tr}_{\boldsymbol{h}} \boldsymbol{U}) h_{ij} = r_{ij} - \tfrac{2}{3}\lambda h_{ij} - S_{ij} - K_i{}^k K_{jk} + K K_{ij}. \tag{3.3.11}$$

In doing so, the principal part now consists of the Ricci curvature operator rather than the Schouten curvature operator. Secondly, we replace the equation $\Lambda^*_i = 0$ with $\bar{\Lambda}_i = 0$, where

$$\bar{\Lambda}_i = D^i \bar{S}_{il} - \epsilon_{ljk} K_i{}^k r^{ij}$$

—this is obtained by rearranging $V_{ij} = 0$ for S_{ij} and substituting into $\Lambda^*_i = 0$. We will call the equations

$$J_{ijk} = 0, \qquad \bar{\Lambda}_i = 0, \qquad \Lambda_i = 0, \qquad V_{ij} = 0,$$

the *Extended Constraint Equations*, or *ECEs* for short, and the map

$$\Psi : \mathscr{S}^2(\mathcal{S}) \times \mathscr{S}^2_0(\mathcal{S}; \boldsymbol{h}) \times \mathscr{S}^2_0(\mathcal{S}; \boldsymbol{h}) \times \mathscr{S}^2(\mathcal{S}) \longrightarrow \mathcal{J}(\mathcal{S}) \times \Lambda^1(\mathcal{S}) \times \Lambda^1(\mathcal{S}) \times \mathscr{S}^2(\mathcal{S}),$$

given by

$$\Psi(\boldsymbol{K}, \bar{\boldsymbol{S}}, \boldsymbol{S}, \boldsymbol{K}) = (J_{ijk},\ \bar{\Lambda}_i,\ \Lambda_i,\ V_{ij}),$$

will be called the *extended constraint map*. From the above discussion it is clear that the ECEs are equivalent to equations (3.3.9a)–(3.3.9d). It can be easily verified that the ECE zero quantities satisfy the following integrability conditions

$$\bar{\Lambda}_l + \tfrac{1}{2}\epsilon_{ijk}D^k J^{ij}{}_l = 0, \tag{3.3.12a}$$
$$\Lambda_j + D_i V_j{}^i - \tfrac{1}{2}D_j V_i{}^i - K_{ik}J_j{}^{ik} + K_{jk}J^{ik}{}_i + KJ_j{}^i{}_i = D^i r_{ij} - \tfrac{1}{2}D_j r = 0. \tag{3.3.12b}$$

Note that the latter essentially encodes the contracted Bianchi identity on $(\mathcal{S}, \boldsymbol{h})$. In identities (3.3.12a) and (3.3.12b) there is a semi-decoupling of the zero quantities J_{ijk} and V_{ij}, which will prove convenient in Chapter 4. This is in contrast to (3.3.10a) and (3.3.10b), in which the J_{ijk} and U_{ij} zero quantities are coupled.

Note also that the traces $V_i{}^i = 0$ and $J^k{}_{ik} = 0$ are precisely the Einstein constraint equations. It follows that any solution of the ECEs implies a solution to the Einstein constraint equations. The reverse is also true, since, having obtained a solution $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$ of the Einstein constraints, one simply *defines*

$$S_{ij} = r_{ij} - \tfrac{2}{3}\lambda h_{ij} - K_i{}^k K_{jk} + KK_{ij}, \tag{3.3.13a}$$
$$\bar{S}_{kl} = -\epsilon_{lij}D^j K_k{}^i. \tag{3.3.13b}$$

By construction then we have $J_{ijk} = 0$, $V_{ij} = 0$, whence the integrability conditions imply that, automatically, $\Lambda_i = \bar{\Lambda}_i = 0$. In other words, the electric and magnetic parts of the Weyl curvature are determined by the initial data, as is well known. Hence, solutions of the ECEs and of the Einstein constraint equations are in one-to-one correspondence. Note that relations (3.3.13a) is equivalent to (3.2.7) and that $\bar{S}_{ij}$ (as given by (3.3.13b)) is automatically symmetric as a consequence of the momentum constraint, since

$$\begin{aligned}
-\epsilon^{kl}{}_m\bar{S}_{kl} &= \epsilon^{kl}{}_m\epsilon_{lij}D^j K_k{}^i \\
&= (h_{im}\delta_j{}^k - h_{jm}\delta_i{}^k)D^j K_k{}^i \\
&= D^j K_{jm} - D_m K.
\end{aligned}$$

3.3.3 Butscher's construction of initial data as perturbations of $(\mathbb{R}^3, \boldsymbol{\delta}, \boldsymbol{0})$

Much of this thesis will be concerned with developing a perturbative method (the *Friedrich–Butscher method*) for solving the ECEs. The method, which was suggested by H. Friedrich and first implemented by A. Butscher in [28, 29] is motivated by the desire to construct regular hyperboloidal solutions of the CCEs. A hyperboloidal initial data set which is a sufficiently small perturbation of a hyperboloid of Minkwoski has a future asymptotically simple (and hence null geodesically complete) spacetime development, by the semi-global existence result of [35].

Ideally, one would like to extend the Friedrich–Butscher method to the full CCEs on manifolds with boundary in order to obtain solutions which are, by construction, regular all the way up to $\partial\mathcal{S} = \mathcal{S} \cap \mathscr{I}$. This method should be contrasted with that of [50], which we discussed briefly at the end of Section 3.3.1, in which establishing regularity (i.e. smoothness) up to $\partial\mathcal{S}$ is a non-trivial task.

As a first step in this direction, A. Butscher takes the "$t = 0$" slice of Minkowski space (in standard coordinates) —i.e. flat initial data $(\mathbb{R}^3, \boldsymbol{\delta}, \boldsymbol{0})$— and proves the existence of solutions of the ECEs as non-linear perturbations of this "background solution". One of the interesting features of this construction is that certain components of the electric and magnetic parts of the Weyl curvaure are prescribed as free data. More precisely (see [29]),

Theorem. (Butscher) Given $k \geq 4$ and $\beta \in (-1, 0)$, there exists a smooth map of weighted Sobolev spaces[3]

$$\begin{aligned}\nu : H^{k-1}_{\beta-1}(\mathscr{C}(\mathbb{R}^3)) \times H^{k-2}_{\beta-2}(\mathscr{S}^2_{TT}(\mathbb{R}^3;\boldsymbol{\delta})) \times H^{k-2}_{\beta-2}(\mathscr{S}^2_{TT}(\mathbb{R}^3;\boldsymbol{\delta}))\\ \longrightarrow H^{k-1}_{\beta-1}(\mathscr{S}^2_0(\mathbb{R}^3;\boldsymbol{\delta})) \times H^{k-1}_{\beta-1}(\Lambda^1(\mathbb{R}^3)) \times H^{k-1}_{\beta-1}(\Lambda^1(\mathbb{R}^3)) \times H^k_\beta(\mathscr{S}^2(\mathbb{R}^3)),\end{aligned}$$

such that $\nu(0, \mathbf{0}, \mathbf{0}) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \boldsymbol{\delta})$ and such that, for sufficiently small input $(\phi, \bar{\boldsymbol{T}}, \boldsymbol{T})$, the output fields $(\boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}, \boldsymbol{h}) \equiv \nu(\phi, \bar{\boldsymbol{T}}, \boldsymbol{T})$ give rise to a solution of the ECEs of the form

$$\begin{aligned}K_{ij} &= \chi_{ij} + \tfrac{1}{3}\phi\delta_{ij},\\ \bar{S}_{ij} &= L(\bar{\boldsymbol{X}})_{ij} + \bar{T}_{ij} - \tfrac{1}{3}(\mathrm{tr}_{\boldsymbol{h}}\bar{\boldsymbol{T}})h_{ij},\\ S_{ij} &= L(\boldsymbol{X})_{ij} + T_{ij} - \tfrac{1}{3}(\mathrm{tr}_{\boldsymbol{h}}\bar{\boldsymbol{T}})h_{ij}\end{aligned}$$

—i.e. *non-linear perturbative solutions* of the ECEs around flat initial data $(\mathbb{R}^3, \boldsymbol{\delta}, \mathbf{0})$. Here, L again denotes the conformal Killing operator (see Section 4.1.1).

Somewhat counter to intuition, perhaps, the case of flat initial data comes with various complications. This is on account of the fact that Minkowski space admits Killing vectors which are asymptotically constant and which arise as *obstructions* to integrability of linearised equations. By an obstruction, we mean an element of the cokernel of the relevant linearised operators, which obstructs the application of the IFT. In addition, flat initial data admits a family of non-trivial *tracefree Codazzi tensors* —see Section 4.1.4 — which also obstruct integrability.

In this thesis, we will restrict instead to closed hypersurfaces, $\mathcal{S}$, in order to put emphasis on the relevant structural features of the ECEs and the identification of potential obstructions. The analysis in the closed case is more delicate since, in particular, one cannot hope to eliminate kernels via the prescription of boundary/decay conditions.

In order to apply the IFT, one first derives from the extended constraint equations a system of so-called *auxiliary* equations which, given the appropriate choice of free and determined data, has a linearisation which is manifestly elliptic. This is done by a combination of identifying an appropriate set of freely prescribed and determined fields, and by a gauge-reduction procedure. For the gauge-reduction, one approach is to use harmonic coordinates in a way that is analogous to the hyperbolic reduction of the Einstein field equations (see Section 1.2) —this is the approach adopted in [28, 29]. A more geometric approach, which we will follow in this thesis, is to use the *De Turck trick*. By construction, any solution of the extended constraint equations will also be a solution of the auxiliary equations. However, having constructed a solution to the auxiliary system —a *candidate initial data set*— it is a non-trivial task to show that it is indeed a solution of the extended constraints. We refer to this as the problem of *sufficiency of the auxiliary system*.

[3] A definition of the weighted Sobolev spaces is given in Chapter 8.

Chapter 4

The Friedrich–Butscher method on closed manifolds: a first application

The purpose of this chapter is to describe a variant of the method given in [28, 29], in the context of closed initial hypersurfaces $\mathcal{S}$. We will refer to this as the Friedrich–Butscher method. As a first application we construct solutions of the ECEs as non-linear perturbations of a class of umbilical background initial data which we refer to as *conformally-rigid, hyperbolic*. These background initial data sets can be thought of as describing initial conditions for spatially-compact analogues of the "$k = -1$" Friedmann–Lemaître–Robinson–Walker (FLRW) spacetime —see [51]. While intended mainly as a proof of concept, to be generalised in Chapter 5, such initial data sets are particularly well-suited to the Friedrich–Butscher method due to the fact that they admit an explicit parametrisation of the free data via the *Gasqui–Goldschmidt–Beig complex*, as a consequence of their conformal flatness.

The work of this chapter is based on the following article:

- Valiente Kroon, J.A. and Williams, J.L., "A perturbative approach to the construction of initial data on compact manifolds", *arXiv preprint*, arXiv:1801.07289 (2018), (Submitted to *Communications in Mathematical Physics*).

The main result, the precise statement of which is given in Theorem 2 of Section 4.3, can be summarised as follows:

Theorem. Let $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$ be a conformally rigid hyperbolic initial data set. Then for each pair of sufficiently small transverse-tracefree (with respect to $\mathring{\boldsymbol{h}}$) tensor fields $T_{ij}, \bar{T}_{ij}$ over $\mathcal{S}$, and each sufficiently small scalar field ϕ over $\mathcal{S}$, there exists a solution of the Einstein constraint equations $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$ with $tr_{\mathring{\boldsymbol{h}}}(\boldsymbol{K} - \mathring{\boldsymbol{K}}) = \phi$ and for which the electric and magnetic parts of the Weyl curvature (restricted to $\mathcal{S}$) of the resulting spacetime development take the form

$$S_{ij} = \mathring{L}(\boldsymbol{X})_{ij} + T_{ij} - \tfrac{1}{3}\mathrm{tr}_{\boldsymbol{h}}(\mathring{L}(\boldsymbol{X}) + \boldsymbol{T})h_{ij},$$
$$\bar{S}_{ij} = \mathring{L}(\bar{\boldsymbol{X}})_{ij} + \bar{T}_{ij} - \tfrac{1}{3}\mathrm{tr}_{\boldsymbol{h}}(\mathring{L}(\bar{\boldsymbol{X}}) + \bar{\boldsymbol{T}})h_{ij},$$

for some covectors $X_i,\ \bar{X}_i$ over $\mathcal{S}$, where L is again the conformal Killing operator (see Section 4.1.1).

The structure of this chapter is as follows: in Section 4.1, we describe in general terms the Friedrich-Butscher method; in Section 4.1.2 we outline the general procedure for the reformulation of the extended constraint equations as an elliptic system; the potential obstructions to the implementation of the method are discussed in Section 4.1.4, motivating our subsequent restriction to *conformally rigid hyperbolic* background initial data. In Section 4.3 the method is carried out in this case, the main result being given in Theorem 2 of Section 4.3.1, and proved by means of Propositions 6 and 9 in Sections 4.3.2 and 4.3.3, respectively.

4.1 An outline of the Friedrich–Butscher method

In this section we will describe in quite general terms a slightly modified version of the method used in [28, 29], which was briefly discussed in the previous chapter.

In an effort to make the presentation as simple as possible we will often restrict the discussion to the principal parts of the relevant equations. More detailed expressions (in particular, for the relevant linearised operators) will be given in subsequent chapters; the content of this section will be elaborated in Chapter 5, in which the Friedrich–Butscher method (and variants) will be studied for a more general class of background geometries. In Section 4.1.4 we will restrict to umbilical background data for a first discussion of the obstructions to the implementation of the Friedrich–Butscher method.

$$
\begin{tikzcd}
\text{Initial data } (\mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}}) \arrow[r, "(i)\,:\,\text{Extend to ECE sol.}", "\text{(algebraic)}"'] \arrow[d, "(\phi, \bar{\boldsymbol{T}}, \boldsymbol{T})"'] & \begin{array}{c}(\mathring{\boldsymbol{K}}, \mathring{\bar{\boldsymbol{S}}}, \mathring{\boldsymbol{S}}, \mathring{\boldsymbol{h}}) \\ \text{solving } \Psi = 0\end{array} \arrow[r, "(ii)\,:\,\text{Solve } \widetilde{\Psi} = 0", "u \equiv (\phi, \bar{\boldsymbol{T}}, \boldsymbol{T})"'] \arrow[d] & \begin{array}{c}\nu(u) \equiv \\ (\boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}, \boldsymbol{h})\end{array} \arrow[d, "(iii)"'] \\
\begin{array}{c}\text{Initial data } (\boldsymbol{h}, \boldsymbol{K}), \\ \text{free-data } (\phi, \bar{\boldsymbol{T}}, \boldsymbol{T})\end{array} & \begin{array}{c}(\boldsymbol{K}, \bar{\boldsymbol{S}}, \boldsymbol{S}, \boldsymbol{h}) \\ \text{solving } \Psi = 0\end{array} \arrow[l, "(v)"'] & \begin{array}{c}w(u) \equiv \\ (\boldsymbol{K}, \bar{\boldsymbol{S}}, \boldsymbol{S}, \boldsymbol{h})\end{array} \arrow[l, "(iv)\,:\,\text{``Sufficiency''}"', "\text{prove } \Psi(w(u)) = 0"]
\end{tikzcd}
$$

Figure 4.1: Schematic representation of the Friedrich–Butscher method.

The above schematic is intended to illustrate the main stages of the method. Stage (i) is trivial, consisting of determining the electric and magnetic parts $\mathring{S}_{ij}, \mathring{\bar{S}}_{ij}$ in terms of $(\mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$ by solving the Gauss–Codazzi and Codazzi–Mainardi equations algebraically —see (3.3.13a) and (3.3.13b). Stage (ii) consists of solving the elliptic auxiliary system, $\widetilde{\Psi} = 0$, for the determined fields $(\boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}, \boldsymbol{h})$ perturbatively —we prove that $D\widetilde{\Psi}$ is an isomorphism, so that the IFT guarantees the existence of a map ν such that $\widetilde{\Psi}(\nu(u)) = 0$. Stage (iii) consists simply of substituting the free data u and determined fields $\nu(u)$ into the appropriate ansatz; collectively, stages $(ii) - (iii)$ will be referred to as the *construction of candidate solutions*. Stage (iv) concerns the issue of *sufficiency*: we must show that the candidate solutions constructed in stages $(ii) - (iii)$ are indeed solutions of the ECEs —i.e. that $\Psi(w(u)) = 0$.

Remark 2. There is an analogy to be drawn here with the procedure of *hyperbolic reduction* —see, for example [34].

4.1.1 Preliminaries

Before proceeding further, it will prove convenient to first define the following differential operators, in terms of a given Riemannian metric, $\boldsymbol{h}$, and its associated Levi-Civita connection, D_i:

- $\Pi_{\boldsymbol{h}} : \mathscr{S}^2(\mathcal{S}) \longrightarrow \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$, the *projection of symmetric 2-tensors into the space of symmetric tracefree 2-tensors*, given by
$$\Pi_{\boldsymbol{h}}(\eta)_{ij} \equiv \eta_{ij} - \tfrac{1}{3}\mathrm{tr}_{\boldsymbol{h}}(\boldsymbol{\eta})h_{ij},$$
- $\star : \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h}) \longrightarrow \mathcal{J}(\mathcal{S})$, given by
$$(\star\boldsymbol{\eta})_{ijk} \equiv \epsilon_{ij}{}^{l}\eta_{kl},$$
where ϵ_{ijk} denotes the volume form;
- δ defined on the space of symmetric valence-κ tensors as follows
$$\delta(\boldsymbol{\eta})_{j_2\cdots j_\kappa} = D^{j_1}\eta_{j_1 j_2\cdots j_\kappa},$$
and $L : \Lambda^1(\mathcal{S}) \longrightarrow \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$ the conformal Killing operator,
$$L(\boldsymbol{X})_{ij} = D_i X_j + D_j X_i - \tfrac{2}{3}D^k X_k h_{ij},$$
- $\mathcal{D} : \mathscr{S}^2(\mathcal{S}) \longrightarrow \mathcal{J}(\mathcal{S})$, the *Codazzi* operator
$$\mathcal{D}(\boldsymbol{\eta})_{ijk} = D_i\eta_{jk} - D_j\eta_{ik},$$
$\mathcal{D}^* : \mathcal{J}(\mathcal{S}) \longrightarrow \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$, the formal L^2-adjoint of $\mathcal{D}$ *restricted* to $\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$,
$$\mathcal{D}^*(\boldsymbol{\mu})_{ij} = D^k\mu_{ikj} + D^k\mu_{jki} - \tfrac{2}{3}D^k\mu_{lk}{}^{l}h_{ij};$$
- $\mathrm{curl} : \Lambda^1(\mathcal{S}) \to \Lambda^1(\mathcal{S})$, defined by
$$\mathrm{curl}(\boldsymbol{X})_i = \epsilon^{jk}{}_i D_j X_k,$$
- $\mathcal{R} : \mathscr{S}(\mathcal{S}) \to \mathscr{S}_0(\mathcal{S};\boldsymbol{h})$, the *rotation* operator
$$\mathcal{R}(\boldsymbol{\eta})_{ij} \equiv \epsilon_{kl(i}D^k\eta^l{}_{j)},$$
- $P_L : \mathscr{S}^2(\mathcal{S}) \longrightarrow \mathscr{S}^2(\mathcal{S})$, the *shifted Lichnerowicz Laplacian*, defined by $P_L = \Delta_L - \frac{2}{3}r$, with Δ_L the *Lichnerowicz Laplacian*, acting as
$$\Delta_L\eta_{ij} \equiv -\Delta\eta_{ij} + 2r_{(i}{}^{k}\eta_{j)k} - 2r_{ikjl}\eta^{kl},$$
- $\Delta_Y : \Lambda^1(\mathcal{S}) \longrightarrow \Lambda^1(\mathcal{S})$, the *Yano Laplacian*, given by
$$\Delta_Y(\boldsymbol{X})_i = -\Delta X_i - r_{ij}X^j,$$
- $\Delta_H : \Lambda^1(\mathcal{S}) \longrightarrow \Lambda^1(\mathcal{S})$, the *Hodge Laplacian*, given by
$$\Delta_H(\boldsymbol{X})_i \equiv \mathrm{curl}^2(\boldsymbol{X}) - d(\delta(\boldsymbol{X})_i = -\Delta X_i + r_{ij}X^j,$$

where, in the above, $\Delta = h^{ij}D_iD_j$, —the "rough Laplacian"— acting on the appropriate tensor spaces. When required, we will indicate the metric with respect to which the operators are defined by appending the metric as a subscript e.g. $\mathcal{D}_{\boldsymbol{h}}$. When defined with respect to a fixed "background metric", $\mathring{\boldsymbol{h}}$, we will add a "$^\circ$" over the kernel letter of the operator.

Remark 3. Since $\mathcal{D} : \mathscr{S}^2(\mathcal{S}) \longrightarrow \mathcal{J}(\mathcal{S})$, the image of $\mathcal{D}$ may be decomposed as in Lemma 1. In particular, given $\eta_{ij} \in \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$, $\mathcal{D}(\boldsymbol{\eta})_{ijk}$ may be decomposed as follows

$$\mathcal{D}(\boldsymbol{\eta})_{ijk} = \tfrac{1}{2}(\epsilon_{ij}{}^{l}\mathcal{R}(\boldsymbol{\eta})_{lk} - \delta(\boldsymbol{\eta})_i h_{jk} + \delta(\boldsymbol{\eta})_j h_{ik}). \tag{4.1.1}$$

It therefore follows that $\mathcal{D}(\boldsymbol{\eta})_{ijk} = 0$ for $\eta_{ij} \in \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$ if and only if $\delta(\boldsymbol{\eta})_i = 0$ *and* $\mathcal{R}(\boldsymbol{\eta})_{ij} = 0$.

We recall that the divergence operator is underdetermined elliptic and (equivalently) the conformal Killing operator L is overdetermined elliptic. Moreover, as shown in [29], the operator $\mathcal{D}$ is overdetermined elliptic when restricted to $\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$. More precisely, one has the following:

Lemma 4. Given a covector $\boldsymbol{\xi}$ let

$$\sigma_{\boldsymbol{\xi}}[\mathcal{D}] : \mathscr{S}^2(\mathcal{S}) \longrightarrow \mathcal{J}(\mathcal{S})$$

denote the symbol map of $\mathcal{D}_{\boldsymbol{h}}$. For $\boldsymbol{\xi} \neq 0$, the kernel of $\sigma_{\boldsymbol{\xi}}[\mathcal{D}]$ is one dimensional, consisting of elements of the form $c\xi_i\xi_j$, $c \in \mathbb{R}$. It follows that the operator $\mathcal{D}|_{\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})}$ is overdetermined elliptic.

Proof. Let $\eta_{ij} \in \mathscr{S}^2(\mathcal{S})$ be in the kernel of the symbol map:

$$\sigma_{\boldsymbol{\xi}}[\mathcal{D}](\boldsymbol{\eta})_{ij} \equiv \xi_i\eta_{jk} - \xi_j\eta_{ik} = 0$$

for $\xi_i \neq 0$. Now, define

$$\eta^0_{ij} \equiv \eta_{ij} - c\xi_i\xi_j,$$

where $c \equiv \frac{\eta}{|\boldsymbol{\xi}|^2}$ with $\eta \equiv \mathrm{tr}_{\boldsymbol{h}}(\boldsymbol{\eta})$, so that

$$\eta_{ij} = \eta^0_{ij} + c\xi_i\xi_j.$$

Note that, by definition, $\eta^0_{ij} \in \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$. Substituting into $\sigma_{\boldsymbol{\xi}}[\mathcal{D}](\boldsymbol{\eta}) = 0$,

$$0 = \xi_i\eta^0_{jk} - \xi_j\eta^0_{ik} + c\xi_i\xi_j\xi_k - c\xi_j\xi_i\xi_k = \xi_i\eta^0_{jk} - \xi_j\eta^0_{ik}. \tag{4.1.2}$$

Tracing with $\boldsymbol{h}$ and using the fact that η^0_{ij} is $\boldsymbol{h}-$tracefree, we see that $\xi^i\eta^0_{ij} = 0$. Contracting (4.1.2) with ξ^i, we obtain

$$0 = |\boldsymbol{\xi}|^2\eta^0_{jk} - \xi^i\xi_j\eta^0_{ik} = |\boldsymbol{\xi}|^2\eta^0_{jk},$$

implying that $\eta^0_{ij} = 0$. Hence,

$$\ker\sigma[\mathcal{D}] = \mathrm{sp}\langle\boldsymbol{\xi}\otimes\boldsymbol{\xi}\rangle.$$

If we restrict $\mathcal{D}$ to act on $\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$ then we see that the symbol map is injective for $\boldsymbol{\xi} \neq 0$, since all non-zero tensors of the form $c\xi_i\xi_j$ have non-zero trace with respect to $\boldsymbol{h}$ —that is to say, $\mathcal{D}|_{\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})}$ is overdetermined elliptic. $\square$

Remark 4. In terms of the above definitions, the extended constraint map is rewritten as follows

$$\Psi(\boldsymbol{K}, \bar{\boldsymbol{S}}, \boldsymbol{S}, \boldsymbol{h}) = \begin{pmatrix} \mathcal{D}_{\boldsymbol{h}}(K)_{ijk} - (\star\bar{S})_{ijk} \\ \delta_{\boldsymbol{h}}(S)_i + \epsilon^{jk}{}_i K_j{}^l \bar{S}_{kl}, \\ \delta_{\boldsymbol{h}}(\bar{S})_i - \epsilon_i{}^{jk} K_k{}^l r_{lj}, \\ \mathrm{Ric}[\boldsymbol{h}]_{ij} - \frac{2}{3}\lambda h_{ij} - S_{ij} + KK_{ij} - K_i{}^k K_{jk} \end{pmatrix}.$$

4.1.2 The auxiliary system and the construction of candidate solutions

As outlined earlier, the Friedrich–Butscher method relies on first using the extended constraint equations to obtain an auxiliary system of equations whose linearisation is elliptic. The existence of solutions is then established by an application of the IFT. In general, the linearised system is a highly coupled second order system of partial differential equations. It will be shown that, in the case of background data with metric of constant sectional curvature (i.e. Einstein manifolds), the linearised equations decouple sufficiently so as to enable a straightforward analysis of its kernel and cokernel —this system will be given in Section 4.3.2. Here, we discuss the procedure in full generality, but for simplicity we restrict attention to the principal parts of the equations, since they suffice for the description of ellipticity.

The ansatz

As remarked above, the operator $\delta_{\boldsymbol{h}}$ is underdetermined elliptic, while $\mathcal{D}_{\boldsymbol{h}}|_{\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})}$ is overdetermined elliptic. Given a *background metric* $\mathring{\boldsymbol{h}}$, the latter observation motivates the following ansatz

$$K_{ij} = \chi_{ij} + \tfrac{1}{3}(\phi + \mathring{K})\mathring{h}_{ij} \tag{4.1.3a}$$
$$S_{ij} = \boldsymbol{S}(\boldsymbol{h}, \boldsymbol{X}, \boldsymbol{T})_{ij} \equiv \Pi_{\boldsymbol{h}}(\mathring{L}(\boldsymbol{X}) + \boldsymbol{T})_{ij} \tag{4.1.3b}$$
$$\bar{S}_{ij} = \bar{\boldsymbol{S}}(\boldsymbol{h}, \bar{\boldsymbol{X}}, \bar{\boldsymbol{T}})_{ij} \equiv \Pi_{\boldsymbol{h}}(\mathring{L}(\bar{\boldsymbol{X}}) + \bar{\boldsymbol{T}})_{ij} \tag{4.1.3c}$$

where $\boldsymbol{\chi}$ is tracefree with respect to the background metric $\mathring{\boldsymbol{h}}$ —i.e. $\boldsymbol{\chi} \in \mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})$; ϕ being the trace part —i.e. $\phi = \mathrm{tr}_{\mathring{\boldsymbol{h}}}\boldsymbol{K}$; and where $\boldsymbol{T}$, $\bar{\boldsymbol{T}}$ are taken to be transverse-tracefree with respect to the background metric —i.e. $\boldsymbol{T}$, $\bar{\boldsymbol{T}} \in \mathscr{S}^2_{TT}(\mathcal{S}, \mathring{\boldsymbol{h}})$. Recall that $\Pi_{\boldsymbol{h}}$ is the projection onto $\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$, so that S_{ij} and $\bar{S}_{ij}$ are tracefree with respect to the metric $\boldsymbol{h}$ (which itself comprises one of the determined fields of the ECEs), as required. The fields

$$u = (\phi, \bar{\boldsymbol{T}}, \boldsymbol{T})$$

will be interpreted as *free data*[1], and

$$v = (\boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}, \boldsymbol{h})$$

will be the *determined fields*. Recalling the definition of the York split —see (2.3.2) of Section 2.3.2— we call (4.1.3b)–(4.1.3c) a *projected York split* of the electric and magnetic tensors.

Expression (4.1.3a) is just the (unique) decomposition into trace and tracefree parts (with respect to $\mathring{\boldsymbol{h}}$) and therefore imposes no restrictions on $\boldsymbol{K}$. Of course, the background solution itself can be

[1] Strictly speaking, the fields $T_{ij}, \bar{T}_{ij}$ are not freely-prescribable since they are required to satisfy $\mathring{\delta}(\bar{\boldsymbol{T}}) = \mathring{\delta}(\boldsymbol{T}) = 0$. A construction for $\bar{T}_{ij}, T_{ij}$, for conformally-flat $\mathring{h}_{ij}$, is discussed in Section 4.4.

expressed in the above form:

$$\mathring{K}_{ij} = \mathring{\chi}_{ij} + \tfrac{1}{3}\mathring{K}\mathring{h}_{ij} \tag{4.1.4a}$$
$$\mathring{S}_{ij} = \mathring{L}(\mathring{\boldsymbol{X}})_{ij} + \mathring{T}_{ij} \tag{4.1.4b}$$
$$\mathring{\bar{S}}_{ij} = \mathring{L}(\mathring{\bar{\boldsymbol{X}}})_{ij} + \mathring{\bar{T}}_{ij}. \tag{4.1.4c}$$

Note that $\mathring{\phi} = 0$, here. Also, since we are taking $\boldsymbol{h} = \mathring{\boldsymbol{h}}$ in this case, the projection operator is just the identity map, $\Pi_{\boldsymbol{h}} = \Pi_{\mathring{\boldsymbol{h}}} = \mathrm{Id}$, and the ansatz for $\mathring{\boldsymbol{S}}$, $\mathring{\bar{\boldsymbol{S}}}$ is simply the York split.

We are adopting a slightly different approach to that of [28, 29], in which the following ansatz is used

$$S_{ij} = L_{\boldsymbol{h}}(\boldsymbol{X})_{ij} + \Pi_{\boldsymbol{h}} T_{ij},$$
$$\bar{S}_{ij} = L_{\boldsymbol{h}}(\bar{\boldsymbol{X}})_{ij} + \Pi_{\boldsymbol{h}} \bar{T}_{ij},$$

with $T_{ij}, \bar{T}_{ij} \in \mathscr{S}^2_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})$ —see Section 3.3.3. The motivation for (4.1.3b)–(4.1.3c) is that the orthogonality property of the York split (with respect to $\mathring{\boldsymbol{h}}$) can be used to argue, in a straightforward way, that the freely-prescribed data $(\phi, \boldsymbol{T}, \bar{\boldsymbol{T}})$ determined their corresponding solutions uniquely. See also Lemma 5, below.

Similarly, (4.1.3b)–(4.1.3c) can also be regarded simply as decompositions, provided $\boldsymbol{h}$ is sufficiently close to $\mathring{\boldsymbol{h}}$ in the $L^\infty-$norm (defined with respect to the background metric, $\mathring{\boldsymbol{h}}$); more precisely, the following holds:

Lemma 5. Suppose that $(\mathcal{S}, \mathring{\boldsymbol{h}})$ admits no conformal Killing vector fields. Then, there exists $\epsilon > 0$ such that, whenever $\|\boldsymbol{h} - \mathring{\boldsymbol{h}}\|_{L^\infty} < \epsilon$, the map

$$\Pi_{\boldsymbol{h}} : \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}}) \longrightarrow \mathscr{S}^2_0(\mathcal{S}; \boldsymbol{h})$$

is an isomorphism, and therefore so is the map

$$\omega : (\phi, \bar{\boldsymbol{T}}, \boldsymbol{T}, \boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}) \mapsto \begin{pmatrix} \chi_{ij} + \frac{1}{3}\phi\mathring{h}_{ij} \\ \bar{\boldsymbol{S}}(\bar{\boldsymbol{X}}, \bar{\boldsymbol{T}})_{ij} \\ \boldsymbol{S}(\boldsymbol{X}, \boldsymbol{T})_{ij} \end{pmatrix}.$$

Proof. First we need to show that $\Pi_{\boldsymbol{h}}$ is injective (for $\boldsymbol{h}$ sufficiently close to $\mathring{\boldsymbol{h}}$ in $\mathcal{B}_{\boldsymbol{h}}$). To see this, note that if $T_{ij} \in \mathrm{Ker}(\ \Pi_{\boldsymbol{h}}) \cap \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})$, then

$$T_{ij} = \tfrac{1}{3} T h_{ij},$$

with $T = \mathrm{tr}_{\boldsymbol{h}}(\boldsymbol{T})$, and

$$0 = T \cdot \mathrm{tr}_{\mathring{\boldsymbol{h}}} \boldsymbol{h} = T \cdot (3 + \mathrm{tr}_{\mathring{\boldsymbol{h}}}(\boldsymbol{h} - \mathring{\boldsymbol{h}})).$$

Now, by Sobolev Embedding (see [43]), the C^0-norm of $(\boldsymbol{h} - \mathring{\boldsymbol{h}})$ is bounded above by the H^2-norm and hence, for $\boldsymbol{h}$ sufficiently close to $\mathring{\boldsymbol{h}}$ in $\mathcal{B}_{\boldsymbol{h}}$, it follows that $T = 0$ and hence $T_{ij} = 0$ —that is to say, $\Pi_{\boldsymbol{h}}$ is injective for such a $\boldsymbol{h}$. In order to show that ω is injective, all that remains to be shown is that the map $u \equiv (\phi, \boldsymbol{T}, \bar{\boldsymbol{T}}) \mapsto \boldsymbol{S}(\boldsymbol{X}(u), \boldsymbol{T})$ is injective (and likewise for $\bar{\boldsymbol{X}}$). The injectivity of the map $u \mapsto \mathring{L}(\boldsymbol{X}(u)) + \boldsymbol{T}$ follows from injectivity of ν and uniqueness of the York split —using, once again, the non-existence of conformal Killing vectors for $\mathring{\boldsymbol{h}}$, see [52]. $\square$

Remark 5. Note that, by Sobolev embedding, $\|\boldsymbol{h}-\mathring{\boldsymbol{h}}\|_{L^\infty}$ is bounded above by $\|\boldsymbol{h}-\mathring{\boldsymbol{h}}\|_{H^2}$. Therefore, the map ω is an isomorphism provided $\|\boldsymbol{h}-\mathring{\boldsymbol{h}}\|_{H^2}$ is sufficiently small. Since the method for solving the ECEs described in this chapter is perturbative, the metric $\boldsymbol{h}$ will, by construction, be automatically close to $\mathring{\boldsymbol{h}}$ in H^2; the norm will in fact be controlled by the appropriate norms of the free data.

We aim to show the existence of solutions, v, of the ECEs corresponding to a given choice of free data, u —i.e. we want to construct a map ν such that

$$v = \omega(\nu(u))$$

is a solution to the ECEs.

The linearisation of the Ricci operator

Let us now consider equation (3.3.9d). As is well known, the linearised Ricci operator is not elliptic; its failure to be so is a consequence of diffeomorphism-invariance, and is connected to the contracted Bianchi identity —see, for instance, [53]. One method of breaking the gauge-invariance is via the use of the so-called *DeTurck trick*. Here we will follow this approach.

Let $\mathring{D}$ denote the Levi-Civita connection associated to $\mathring{\boldsymbol{h}}$. The *linearisation of the Ricci operator* about $\mathring{h}_{ij}$, as a map from symmetric tensor fields to symmetric tensor fields, is given by the following *Fréchet derivative*

$$\begin{aligned} D\mathrm{Ric}[\mathring{\boldsymbol{h}}]\cdot(\boldsymbol{\gamma})_{ij} &\equiv \frac{d}{d\tau} r[\mathring{\boldsymbol{h}}+\tau\boldsymbol{\gamma}]_{ij}\bigg|_{\tau=0} \qquad (4.1.5)\\ &= -\tfrac{1}{2}\mathring{\Delta}\gamma_{ij} + \tfrac{1}{2}\mathring{D}_k\mathring{D}_i\gamma_j{}^k + \tfrac{1}{2}\mathring{D}_k\mathring{D}_j\gamma_i{}^k - \tfrac{1}{2}\mathring{D}_i\mathring{D}_j\gamma \\ &= -\tfrac{1}{2}\mathring{\Delta}\gamma_{ij} + \tfrac{1}{2}\mathring{D}_i\mathring{D}_k\gamma_j{}^k + \tfrac{1}{2}\mathring{D}_j\mathring{D}_k\gamma_i{}^k - \tfrac{1}{2}\mathring{D}_i\mathring{D}_j\gamma + \mathring{r}_{(i}{}^k\gamma_{j)k} - \mathring{r}_{ikjl}\gamma^{kl} \\ &= \tfrac{1}{2}\Delta_L\gamma_{ij} + \mathring{D}_{(i}C(\boldsymbol{\gamma})_{j)}{}^k{}_k, \qquad (4.1.6) \end{aligned}$$

—see [54] for more details. Here, τ should be understood as a parametrisation of a one-parameter-family of metrics, $\boldsymbol{h}(\tau) = \mathring{\boldsymbol{h}} + \tau\boldsymbol{\gamma}$, and for later use, we are defining $C(\cdot)^i{}_{jk}$ by

$$C(\boldsymbol{\gamma})^i{}_{jk} \equiv \tfrac{1}{2}(\mathring{D}_j\gamma_k{}^i + \mathring{D}_k\gamma_j{}^i - \mathring{D}^i\gamma_{jk}) \qquad (4.1.7)$$

—this is precisely the linearisation of the Christoffel symbols. Here, and in what follows, index raising and lowering within a linearised covariant expression will be understood to be with respect to the background metric, $\mathring{h}_{ij}$, and its inverse, $\mathring{h}^{ij}$. The first term of (4.1.6), $\Delta_L\gamma_{ij}$, is manifestly elliptic, but the ellipticity is spoiled by the second-order term $\mathring{D}_{(i}C_{j)}{}^k{}_k$.

Definition 9. Given an arbitrary local coordinate system, (x^α), and two Riemannian metrics $\boldsymbol{h}, \bar{\boldsymbol{h}}$ define the *De Turck vector* as follows

$$Q(\boldsymbol{h};\bar{\boldsymbol{h}})^\alpha = h^{\beta\gamma}(\Gamma[\boldsymbol{h}]^\alpha_{\beta\gamma} - \Gamma[\bar{\boldsymbol{h}}]^\alpha_{\beta\gamma}),$$

where $\Gamma[\boldsymbol{h}]^\alpha_{\beta\gamma}, \Gamma[\bar{\boldsymbol{h}}]^\alpha_{\beta\gamma}$ denote the Christoffel symbols of $\boldsymbol{h}, \bar{\boldsymbol{h}}$, respectively. The metric $\bar{\boldsymbol{h}}$ is thought of as a reference metric.

Remark 6. Note that, though Q^α is defined with respect to a fixed local coordinate system, the expression is in fact covariant as it is constructed from the difference of two connections. Hence, $\boldsymbol{Q}$

is a (globally-defined) vector field, which we will henceforth denote in the abstract index formalism by Q^i. The remaining calculations of the chapter will be carried out in the abstract index notation. Similar to the generalisation of harmonic coordinates to *generalised harmonic coordinates*, the De Turck vector may be generalised by the addition of an arbitrary vector field, $\boldsymbol{V}$:

$$\boldsymbol{Q}(\boldsymbol{h}, \boldsymbol{V}; \mathring{\boldsymbol{h}})^\alpha \equiv (\Gamma[\boldsymbol{h}]^\alpha_{\beta\gamma} - \Gamma[\bar{\boldsymbol{h}}]^\alpha_{\beta\gamma}) - 2V^\alpha.$$

We will return to this idea in Chapter 5.

Given a one-parameter-family of metrics, $\boldsymbol{h}(\tau)$, we define

$$Q(\tau)^\alpha = Q(\boldsymbol{h}(\tau); \mathring{\boldsymbol{h}}) \equiv h(\tau)^{\beta\gamma}(\Gamma(\tau)^\alpha_{\beta\gamma} - \mathring{\Gamma}^\alpha_{\beta\gamma}),$$

where $\Gamma(\tau)^\alpha_{\beta\gamma} = \Gamma[\boldsymbol{h}(\tau)]^\alpha_{\beta\gamma}$ and $\mathring{\Gamma}^\alpha_{\beta\gamma} = \Gamma[\mathring{\boldsymbol{h}}]^\alpha_{\beta\gamma}$. Consider now the Lie derivative of the metric along $Q(\tau)$, $\mathcal{L}_{Q(\tau)} h(\tau)_{ij}$, the linearisation of which is given by

$$\frac{d}{d\tau}(\mathcal{L}_{Q(\tau)}\boldsymbol{h}(\tau))_{ij}\bigg|_{\tau=0} = 2\mathring{D}_{(i}C(\boldsymbol{\gamma})_{j)}{}^k{}_k,$$

which is precisely the term in (4.1.6) obstructing the ellipticity in the linearised Ricci operator. Accordingly, we define the *reduced Ricci operator*, $\widetilde{\mathrm{Ric}}(\cdot)$, as

$$\widetilde{\mathrm{Ric}}[\boldsymbol{h}]_{ij} \equiv \mathrm{Ric}[\boldsymbol{h}]_{ij} - \tfrac{1}{2}(\mathcal{L}_Q h)_{ij}.$$

The linearisation of the reduced Ricci operator can then be seen to be proportional to the Lichnerowicz Laplacian of the background metric —that is,

$$D\widetilde{\mathrm{Ric}}[\mathring{\boldsymbol{h}}] \cdot \gamma_{ij} = \tfrac{1}{2}\mathring{\Delta}_L \gamma_{ij},$$

which is manifestly elliptic —note that, modulo curvature terms, Δ_L is simply the rough Laplacian and, therefore, clearly elliptic —see [54] for an alternative discussion of the above. Although we will not need it, we note that in fact the non-linear operator $\widetilde{\mathrm{Ric}}[\cdot]$ is also (quasi-linear) elliptic, by the following —see [54]:

Proposition. (De Turck/Schoen) Given a fixed Riemannian metric, $\bar{\boldsymbol{h}}$, $\mathrm{Ric}[\boldsymbol{h}]$ admits a global expression of the form

$$\mathrm{Ric}[\boldsymbol{h}]_{ij} = -\tfrac{1}{2}\mathrm{tr}_{\boldsymbol{h}}\bar{D}^2\boldsymbol{h}_{ij} + \tfrac{1}{2}\mathcal{L}_{\boldsymbol{Q}(\boldsymbol{h};\bar{\boldsymbol{h}})}h_{ij} + \mathscr{Q}(\boldsymbol{h}, \bar{D}\boldsymbol{h})_{ij} + \mathrm{Ric}[\bar{\boldsymbol{h}}]_{ij},$$

where $\bar{D}$ denotes the Levi-Civita connection of $\bar{\boldsymbol{h}}$, $Q(\boldsymbol{h}; \mathring{\boldsymbol{h}})^i$ is again the de Turck vector, as above, and $\mathscr{Q}(\boldsymbol{h}, \bar{D}\boldsymbol{h})$ is an expression quadratic in $\bar{D}\boldsymbol{h}$.

Fixing $\bar{\boldsymbol{h}} = \mathring{\boldsymbol{h}}$ in the above, we have

$$\widetilde{\mathrm{Ric}}[\boldsymbol{h}] = -\tfrac{1}{2}\mathrm{tr}_{\boldsymbol{h}}\mathring{D}^2\boldsymbol{h}_{ij} + \mathscr{Q}(\boldsymbol{h}, \mathring{D}\boldsymbol{h})_{ij} + \mathrm{Ric}[\mathring{\boldsymbol{h}}]_{ij},$$

which is quasi-linear second-order elliptic. It is not surprising then that $D\widetilde{\mathrm{Ric}}$ is elliptic.

Remark 7. The reduced Ricci operator coincides with the Ricci operator when $Q^i = 0$. The linearisation $D\widetilde{\mathrm{Ric}}(\cdot)$ is formally identical to that obtained through the use of (generalised) harmonic coordinates. We will sometimes refer to the de Turck trick, or variants thereof, as elliptic *gauge-reductions*. This is somewhat of a misnomer since the de Turck is geometric (i.e. the resulting expressions are tensorial) and does not rely on a choice of coordinates.

The auxiliary extended constraint map

Following the discussion of the previous subsections, it is convenient to define the *auxiliary extended constraint map*

$$\widetilde{\Psi}(\boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}, \boldsymbol{h}; \phi, \bar{\boldsymbol{T}}, \boldsymbol{T}) \equiv \begin{pmatrix} \mathring{\mathcal{D}}^*(\boldsymbol{J})_{ij} \\ \bar{\Lambda}_i \\ \Lambda_i \\ V_{ij} - \frac{1}{2}\mathcal{L}_Q h_{ij} \end{pmatrix} = \begin{pmatrix} \mathring{\mathcal{D}}^* \left(\mathcal{D}_{\boldsymbol{h}}(\boldsymbol{K}) - \star\bar{\boldsymbol{S}}\right)_{ij} \\ \delta_{\boldsymbol{h}}(\bar{\boldsymbol{S}})_i - \epsilon^{jk}{}_i K_j{}^l r_{kl} \\ \delta_{\boldsymbol{h}}(\boldsymbol{S})_i + \epsilon^{jk}{}_i K_j{}^l \bar{S}_{kl} \\ \widetilde{\mathrm{Ric}}[\boldsymbol{h}]_{ij} - \frac{2}{3}\lambda h_{ij} - S_{ij} + K K_{ij} - K_i{}^k K_{jk} \end{pmatrix},$$

with the understanding that the fields $K_{ij},\ S_{ij},\ \bar{S}_{ij}$ should be substituted by the ansatz (4.1.3a)–(4.1.3c). The *auxiliary system* is then given by the vanishing of the *auxiliary extended constraint map*

$$\widetilde{\Psi}(\boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}, \boldsymbol{h}; \phi, \bar{\boldsymbol{T}}, \boldsymbol{T}) = 0, \tag{4.1.8}$$

and should be read as a (second-order) system of partial differential equations for the fields $\boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}, \boldsymbol{h}$, with the fields $\phi, \bar{\boldsymbol{T}}, \boldsymbol{T}$ regarded as input —i.e. they are the freely specifiable data.

Remark 8. Note that the auxiliary system is always defined with reference to some fixed *background* metric $\mathring{\boldsymbol{h}}$ which enters both through the ansatz (4.1.3a)-(4.1.3c) and through the definition of the reduced Ricci operator. Any solution $(\boldsymbol{\chi}, \bar{\boldsymbol{S}}, \boldsymbol{S}, \boldsymbol{h})$ of the extended constraint equations is also a solution of the auxiliary system.

In the following, we denote by $D_u\widetilde{\Psi}[\mathring{\boldsymbol{K}}, \mathring{\bar{\boldsymbol{X}}}, \mathring{\boldsymbol{X}}, \mathring{\boldsymbol{h}}]$ and $D_v\widetilde{\Psi}[\mathring{\boldsymbol{K}}, \mathring{\bar{\boldsymbol{X}}}, \mathring{\boldsymbol{X}}, \mathring{\boldsymbol{h}}]$ the linearisations of $\widetilde{\Psi}$, at $(\mathring{\boldsymbol{K}}, \mathring{\bar{\boldsymbol{X}}}, \mathring{\boldsymbol{X}}, \mathring{\boldsymbol{h}})$, in the directions of the free and determined fields, respectively —that is to say, the following linear maps

$$D_u\widetilde{\Psi}[\mathring{\boldsymbol{K}}, \mathring{\bar{\boldsymbol{X}}}, \mathring{\boldsymbol{X}}, \mathring{\boldsymbol{h}}] \cdot (\breve{\phi}, \breve{\bar{\boldsymbol{T}}}, \breve{\boldsymbol{T}}) = \frac{d}{d\tau}\widetilde{\Psi}(\mathring{\boldsymbol{\chi}}, \mathring{\boldsymbol{X}}, \mathring{\bar{\boldsymbol{X}}}, \mathring{\boldsymbol{h}};\ \mathring{\phi} + \tau\breve{\phi}, \mathring{\bar{\boldsymbol{T}}} + \tau\breve{\bar{\boldsymbol{T}}}, \mathring{\boldsymbol{T}} + \tau\breve{\boldsymbol{T}})\bigg|_{\tau=0}, \tag{4.1.9a}$$

$$D_v\widetilde{\Psi}[\mathring{\boldsymbol{K}}, \mathring{\bar{\boldsymbol{X}}}, \mathring{\boldsymbol{X}}, \mathring{\boldsymbol{h}}] \cdot (\boldsymbol{\sigma}, \boldsymbol{\xi}, \bar{\boldsymbol{\xi}}, \gamma) = \frac{d}{d\tau}\widetilde{\Psi}(\mathring{\boldsymbol{\chi}} + \tau\boldsymbol{\sigma}, \mathring{\boldsymbol{X}} + \tau\boldsymbol{\xi}, \mathring{\bar{\boldsymbol{X}}} + \tau\bar{\boldsymbol{\xi}}, \mathring{\boldsymbol{h}} + \tau\boldsymbol{\gamma};\ \phi, \bar{\boldsymbol{T}}, \boldsymbol{T})\bigg|_{\tau=0}, \tag{4.1.9b}$$

where $\mathring{\boldsymbol{X}},\ \mathring{\bar{\boldsymbol{X}}},\ \mathring{\boldsymbol{S}},\ \mathring{\bar{\boldsymbol{S}}}$, and $\mathring{\boldsymbol{\chi}}$ are as given in (4.1.4a)–(4.1.4c). The explicit expressions will be given in the next chapter.

Notation. We will often suppress the dependence on the background solution and denote the linearisations simply by $D_u\widetilde{\Psi}$ and $D_v\widetilde{\Psi}$ for notational convenience.

Note that, as they are held fixed, the free data $(\phi, \bar{\boldsymbol{T}}, \boldsymbol{T})$ are not an input for $D_v\widetilde{\Psi}$. We will not give the expression for $D_v\widetilde{\Psi}$ for a general background here; this will be deferred to the following chapter. It will suffice for the purposes of this section to consider only the principal parts of $D_v\widetilde{\Psi} = 0$ as a second-order system of partial differential equations —namely,

$$\begin{pmatrix} \mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}} & \mathring{\mathcal{D}}^*(\mathring{\star}\mathring{L}) & 0 & 0 \\ 0 & \mathring{\delta} \circ \mathring{L} & 0 & 0 \\ 0 & 0 & \mathring{\delta} \circ \mathring{L} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\mathring{\Delta} \end{pmatrix} \begin{pmatrix} \sigma_{ij} \\ \bar{\xi}_i \\ \xi_i \\ \gamma_{ij} \end{pmatrix}.$$

Since the principal part is upper-triangular, to verify ellipticity of the full system we need consider only the diagonal entries, which are elliptic by construction —one proceeds from the bottom-right, verifying invertibility of the symbol of each row, and successively substituting into the row above where necessary. It follows then that, for smooth background fields $(\mathring{\boldsymbol{K}}, \mathring{\bar{\boldsymbol{X}}}, \mathring{\boldsymbol{X}}, \mathring{\boldsymbol{h}})$ over a compact $\mathcal{S}$, the operator $D_v\widetilde{\Psi}$ is Fredholm; in particular, it satisfies the Fredholm alternative —see Section 2.3.2.

If $D_v\widetilde{\Psi}$ can be shown to be an isomorphism (on the appropriate choice of Banach spaces), one can then proceed to apply the IFT in order to construct solutions to $\widetilde{\Psi} = 0$ —one would obtain a map ν such that $\widetilde{\Psi}(\nu(u); u) = 0$.

Remark 9. Note that the proof of the IFT, which is based on successive approximations of the solution via the linearised operator, provides an effective means of construction of the actual solution —i.e. the method is *constructive*, in contrast to methods using, say, the Schauder fixed-point Theorem.

4.1.3 The sufficiency argument

Let us now assume that *Step (i)* has been carried out: that is to say, that we have established the existence of a small neighbourhood of solutions to the auxiliary system (4.1.8). In particular we have

$$\mathring{\mathcal{D}}^*(J)_{ij} = 0, \tag{4.1.10a}$$
$$V_{ij} = \tfrac{1}{2}(\mathcal{L}_Q\boldsymbol{h})_{ij}, \tag{4.1.10b}$$
$$\Lambda_i = \bar{\Lambda}_i = 0. \tag{4.1.10c}$$

In order to conclude that the solutions of the auxiliary system indeed solve the extended constraint equations, there remains the task of showing:

(a) that $(\mathcal{L}_Q\boldsymbol{h})_{ij} = 0$ —i.e. that $\mathrm{Ric}[\boldsymbol{h}] = \widetilde{\mathrm{Ric}}[\boldsymbol{h}]$— in order that equation (3.3.9d), namely $V_{ij} = 0$, holds;

(b) that $J_{ijk} = 0$ so that (3.3.9a) is satisfied.

Remark 10. Item (a) can be thought of as the analogue of gauge propagation in the hyperbolic reduction of the Einstein field equations.

Tasks (a)–(b) will be carried out with the help of the integrability conditions (3.3.12a)–(3.3.12b), which in view of (4.1.10c), reduce to

$$\epsilon^{ijk}D_iJ_{jkl} = 0, \tag{4.1.11a}$$
$$D^i(\mathcal{L}_Q\boldsymbol{h})_{ij} - \tfrac{1}{2}D_j(\mathcal{L}_Q\boldsymbol{h})_i{}^i = 2K_{ik}J_j{}^{ik} - 2K_{jk}J^{ik}{}_i - 2KJ_j{}^i{}_i. \tag{4.1.11b}$$

The strategy will be to use (4.1.10a) and (4.1.11a) to first show that $J_{ijk} = 0$, and then to substitute into (4.1.11b), which will be used to show $Q_i = 0$.

Identities for Q_i and J_{ijk}

In the forthcoming sections we outline the derivation of two integral identities, (4.1.12) and (4.1.16), that will form the basis of the sufficiency argument. We follow [28, 29] —while the derivations are fundamentally the same, we take care to keep track of the various curvature quantities that arise, the detailed knowledge of which was not required for the purposes of [28, 29]. On the other hand, since we are dealing here with a closed manifold $\mathcal{S}$, there are no boundary terms to keep track of when integrating by parts —this is not the case in [28, 29].

We begin by noticing that

$$\begin{aligned} D^i(\mathcal{L}_Q\boldsymbol{h})_{ij} - \tfrac{1}{2}D_j(\mathcal{L}_Q\boldsymbol{h})_i{}^i &= D^i\left(D_iQ_j + D_jQ_i - D^kQ_kh_{ij}\right) \\ &= \Delta Q_j + D^iD_jQ_i - D_jD^kQ_k \\ &= \Delta Q_j + r_{ij}Q^i \\ &= -\Delta_Y Q_j. \end{aligned}$$

Therefore, if $J_{ijk}=0$, then (4.1.11b) implies, after integration by parts over the closed manifold $\mathcal{S}$, that

$$\int_{\mathcal{S}} \left(\|D\boldsymbol{Q}\|^2_{\boldsymbol{h}} - r_{ij}Q^iQ^j\right)\, d\mu_{\boldsymbol{h}} = 0. \tag{4.1.12}$$

Note that the above identity only follows once it has been established that $J_{ijk}=0$. In the next section an integral identity given solely in terms of J_{ijk} will be derived —this is a consequence of the semi-decoupling of (4.1.11b)–(4.1.11a)— allowing for a two step approach in which we first show $J_{ijk}=0$ and then use (4.1.12) to show $Q_i=0$.

A similar identity may be derived for the zero-quantity J_{ijk}. We follow here the derivation in [29], omitting some of the details.

Using decomposition (2.1.9), equations (4.1.10a) and (4.1.11a) may be rewritten as

$$\epsilon_{kjl}D^lF_i{}^j + \epsilon_{ijl}D^lF_k{}^j - L_h(A)_{ik} = 0 \tag{4.1.13a}$$

$$D_iF_l{}^i - \epsilon_{lij}D^jA^i = 0 \tag{4.1.13b}$$

We will see later that this system is, in fact, a first-order elliptic system for the components $A_i,\ F_{ij}$. This will form the foundation for the "sufficiency argument" of Section 4.3.3.

Now, by virtue of F_{ij} being tracefree, it follows that

$$\epsilon^l{}_{ij}F_{kl} + \epsilon^l{}_{jk}F_{il} + \epsilon^l{}_{ki}F_{jl} = 0$$

—i.e. $\epsilon^l{}_{ij}F_{kl}$ has the Jacobi property. Taking the divergence of the previous expression and combining with (4.1.13b), one obtains

$$\epsilon_{kjl}D^lF_i{}^j = -D_iA_k + D_kA_i + \epsilon_{ijl}D^lF_k{}^j.$$

The latter, in turn, when substituted into (4.1.13a) gives

$$D_nF_{im} - D_mF_{in} - \epsilon_{mnj}D^jA_i + \tfrac{1}{3}\epsilon_{imn}D_jA^j = 0.$$

Taking the divergence once more one finds that

$$\Delta_{\boldsymbol{h}}F_{im} - \epsilon_{ijn}D_mD^nA^j + \tfrac{1}{3}\epsilon_{imn}D^nD_jA^j - F_i{}^jr_{mj} + F^{jn}r_{ijmn} - \tfrac{1}{2}A^j\epsilon_m{}^{nk}r_{ijnk} = 0,$$

where we have again used (4.1.13b). Decomposing the above equation into its symmetric and antisymmetric parts one obtains the system

$$\Delta_{\boldsymbol{h}} F_{im} - \epsilon_{jn(m} D_{i)} D^n A^j - r_{j(i} F_{m)}{}^j + F^{jn} r_{ijmn} - \tfrac{1}{2} A^j \epsilon_{(m}{}^{nk} r_{i)jnk} = 0, \tag{4.1.14a}$$

$$\Delta_{\boldsymbol{h}} A_i - \tfrac{1}{3} D_i D^k A_k - F^{kj} \epsilon_{ijm} r_k{}^m = 0. \tag{4.1.14b}$$

Contracting with F^{im} and A^i, respectively, and integrating by parts, one obtains after a lengthy calculation —see [29] for more details— the identities

$$\int_{\mathcal{S}} \left(\|D\boldsymbol{F}\|^2_{\boldsymbol{h}} - \|D\boldsymbol{A}\|^2_{\boldsymbol{h}} + |\delta_{\boldsymbol{h}} \boldsymbol{A}|^2 + \mathcal{R}_1(\boldsymbol{A}, \boldsymbol{F}) \right) d\mu_{\boldsymbol{h}} = 0, \tag{4.1.15a}$$

$$\int_{\mathcal{S}} \left(\|D\boldsymbol{A}\|^2_{\boldsymbol{h}} - \tfrac{1}{3} |\delta_{\boldsymbol{h}} \boldsymbol{A}|^2 + \mathcal{R}_2(\boldsymbol{A}, \boldsymbol{F}) \right) d\mu_{\boldsymbol{h}} = 0, \tag{4.1.15b}$$

where $\mathcal{R}_1(\boldsymbol{A}, \boldsymbol{F})$ and $\mathcal{R}_2(\boldsymbol{A}, \boldsymbol{F})$ are given by

$$\mathcal{R}_1(\boldsymbol{A}, \boldsymbol{F}) \equiv -r^{ij} A_i A_j + r_{jm} F_i{}^m F^{ij} - r_{imjn} F^{ij} F^{mn} - \tfrac{1}{2} \epsilon_j{}^{nk} r_{imnk} F^{jm} A^i,$$

$$\mathcal{R}_2(\boldsymbol{A}, \boldsymbol{F}) \equiv \epsilon_{imp} r_j{}^p F^{jm} A^i.$$

Adding three times (4.1.15b) to (4.1.15a), one obtains

$$\int_{\mathcal{S}} \left(\|D\boldsymbol{F}\|^2_{\boldsymbol{h}} + 2\|D\boldsymbol{A}\|^2_{\boldsymbol{h}} + \mathcal{R}(\boldsymbol{A}, \boldsymbol{F}) \right) d\mu_{\boldsymbol{h}} = 0, \tag{4.1.16}$$

where $\mathcal{R}(\boldsymbol{A}, \boldsymbol{F})$ denotes the quadratic form on A_i, F_{ij} given by

$$\mathcal{R}(\boldsymbol{A}, \boldsymbol{F}) \equiv -r_{ij} A^i A^j + r_{jm} F_i{}^m F^{ij} + 3\epsilon_{imn} r_j{}^n F^{jm} A^i - r_{imjn} F^{ij} F^{mn} - \tfrac{1}{2} \epsilon_j{}^{np} r_{imnp} F^{jm} A^i.$$

The idea of the sufficiency argument is to establish positivity of the integrands in (4.1.12) and (4.1.16) in order to show that $Q_i,\ J_{ijk}$ must necessarily vanish, at least for solutions to the auxiliary system sufficiently close to the background solution. The detailed argument for the case of *hyperbolic background initial data* will be given in Section 4.3.3.

4.1.4 Obstructions to the existence of solutions: first considerations

In order to use the IFT (see Section 4.3.2) to establish existence of solutions to the auxiliary system

$$\widetilde{\Psi} = 0,$$

one would like to show that the linearisation $D_v \widetilde{\Psi}$ is an isomorphism between suitable Banach spaces. Accordingly, by an *obstruction to the existence of solutions*, we mean a non-trivial element of either $\ker(D_v \widetilde{\Psi})$ or $\mathrm{coker}(D_v \widetilde{\Psi})$ —recalling that $D_v \widetilde{\Psi}$ is an elliptic operator (and hence Fredholm for compact $\mathcal{S}$), the existence of a non-trivial cokernel is precisely the obstruction to surjectivity of $D_v \widetilde{\Psi}$ while the existence of a non-trivial kernel is the obstruction to injectivity. Since the space of potential obstructions depends on the structure of the auxiliary system —i.e. on our choice of elliptic reduction— a systematic study will not be given, here.

As it will be seen, among the potential obstructions to the existence of solutions one has non-trivial conformal Killing vectors and tracefree Codazzi tensors of the background manifold. Precluding the existence of such obstructions is the fundamental motivation behind our choice of background data.

Remark 11. It is not clear whether the obstructions, identified in the following section, can be circumvented. In [28, 29], for instance, the method follows through despite the existence of non-trivial conformal Killing vectors. There, in *Step* (*i*), the auxiliary system is solved only up to an *error term*, constrained to lie in a finite-dimensional space. In *Step* (*ii*), it is then simultaneously shown that the error term must necessarily vanish and that the extended constraints are indeed satisfied, as a consequence of the non-linear integrability conditions (4.1.11a)-(4.1.11b). Whether such a procedure may be implemented in general is unclear. One might expect the method to be more rigid in the compact case —the non-existence of conformal Killing vectors, for instance, may be a prerequisite. An analogy may be drawn here with the problem of *linearisation stability* of the constraint equations, in which the obstructions to integrability are precisely the so-called *KID sets*, describing the projection onto $\mathcal{S}$ of a spacetime Killing vector. In the case of non-compact $\mathcal{S}$, a solution of the constraint equations may still be linearisation stable even when it admits a KID set, at least when the perturbations of the initial data are restricted to those of sufficiently fast decay at infinity (see for example [55]), while the compact case is more rigid. On the other hand, it may be the case that the obstructions can be circumvented altogether using a more involved argument, as in [28, 29].

Conformal Killing vectors

It is clear from the construction of the auxiliary system that the existence of a non-trivial conformal Killing vector in the background Riemannian manifold $(\mathcal{S}, \mathring{\boldsymbol{h}})$, η_i say, destroys the injectivity of $D_v\widetilde{\Psi}$, because of the use of the ansatz (4.1.3b)-(4.1.3c). Indeed, $\ker(D_v\widetilde{\Psi})$ contains linear combinations of

$$(\sigma_{ij},\ \bar{\xi}_i,\ \xi_i,\ \gamma_{ij}) = (\mathbf{0},\ \eta_i,\ \mathbf{0},\ \mathbf{0}) \qquad \text{and} \qquad (\sigma_{ij},\ \bar{\xi}_i,\ \xi_i,\ \gamma_{ij}) = (\mathbf{0},\ \mathbf{0},\ \eta_i,\ \mathbf{0}).$$

Moreover, in the case of a constant mean curvature background, the second component of $D_v\widetilde{\Psi}$ takes the form

$$\mathring{\delta}(\mathring{L}(\bar{\boldsymbol{\xi}})) = 0$$

and therefore in this case $\operatorname{coker}(D_v\widetilde{\Psi})$ also contains elements of the form

$$(\sigma_{ij},\ \bar{\xi}_i,\ \xi_i,\ \gamma_{ij})^* = (\mathbf{0},\ \eta_i,\ \mathbf{0},\ \mathbf{0}),$$

so that $D_v\widetilde{\Psi}$ also fails to be surjective —here we are using the suffix * as a shorthand to denote an arbitrary element of the codomain of $D_v\widetilde{\Psi}$. Similar difficulties arise in both the conformal method and the gluing methods, whenever there exist non-trivial conformal Killing vectors —see, for instance, [7].

Remark 12. From the previous discussion, it follows that the implementation of the Friedrich–Butscher method will be simplified if one restricts to *background initial data sets which do not admit a conformal Killing vector*. This condition holds, in particular, for manifolds of negative-definite Ricci curvature. To see this, note that the conformal Killing equation implies after contraction with $D^i\eta^j$ and integration by parts that

$$\int_{\mathcal{S}} \left(\|\mathring{D}\boldsymbol{\eta}\|^2_{\mathring{\boldsymbol{h}}} + \tfrac{1}{3}|\mathring{\delta}(\boldsymbol{\eta})|^2 - \mathring{r}_{ij}\eta^i\eta^j \right)\ d\mu_{\mathring{\boldsymbol{h}}} = 0.$$

Thus, if the Ricci tensor is negative-definite then $\eta_i = 0$ as a consequence of the positive-definiteness of the integrand. This is valid in particular for Einstein metrics of negative scalar curvature, despite them being locally maximally-symmetric —that is to say that, while there exists the maximal number of *local* Killing vector fields in a neighbourhood of each point, none may be extended globally to the whole manifold. A sufficient condition for the (stronger) requirement of non-existence of *local* conformal Killing vector fields is given in [56].

Non-trivial tracefree Codazzi tensors

Inspection of the auxiliary equation for the extrinsic curvature, $\mathring{\mathcal{D}}^*(\boldsymbol{J})_{ij} = 0$, readily shows that the existence of non-trivial tracefree *Codazzi* tensors in the background initial data set —i.e. elements of $\ker(\mathring{\mathcal{D}}) \cap \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})$— also give rise to obstructions similar in nature to those arising from the existence of conformal Killing vectors. In this case, given a tracefree Codazzi tensor, η_{ij} say, $\ker(D_v\widetilde{\Psi})$ and $\operatorname{coker}(D_v\widetilde{\Psi})$ both contain elements of the form

$$(\eta_{ij},\ \mathbf{0},\ \mathbf{0},\ \mathbf{0}),$$

which destroy both the injectivity and the surjectivity of $D_v\widetilde{\Psi}$.

For examples of initial data sets which *do* admit tracefree Codazzi tensors, consider manifolds $(\mathcal{S}, \mathring{\boldsymbol{h}})$ of *harmonic curvature* —that is to say (see [42]) such that

$$\mathring{D}^l \mathring{r}_{ijkl} = 0,$$

or equivalently (using the Kulkarni–Nomizu decomposition), such that

$$\mathring{\mathcal{D}}(\mathring{\boldsymbol{r}})_{ijk} = \mathring{D}_i \mathring{r}_{jk} - \mathring{D}_j \mathring{r}_{ik} = 0.$$

Contracting the above and using the contracted Bianchi identity, it follows that for such a metric $\mathring{r}$ is constant and hence that the tracefree Ricci curvature, denoted here by $\mathring{d}_{ij}$, is a tracefree Codazzi tensor which is moreover non-trivial unless $\mathring{\boldsymbol{h}}$ is Einstein. Since the scalar curvature is constant, it is also straightforward to see that for any choice of real constant $\mathring{K}$, $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}} = \frac{1}{3}\mathring{K}\mathring{\boldsymbol{h}})$ constitutes an umbilical initial data set with cosmological constant

$$\lambda = \tfrac{1}{2}\mathring{r} + \tfrac{1}{3}\mathring{K}^2.$$

Note that by the Weyl–Schouten Theorem (see Theorem 5.1 in [34]), a metric $\mathring{\boldsymbol{h}}$ of harmonic curvature is (locally) conformally flat, since the Cotton–York tensor vanishes[2]:

$$\mathcal{H}_{ij} \equiv \mathring{\epsilon}_{kl(i}\mathring{D}^k \mathring{l}_{j)}{}^l \equiv \mathring{\epsilon}_{kl(i}\mathring{D}^k \mathring{r}_{j)}{}^l = \tfrac{1}{2}\mathring{\epsilon}_{kl(i}\mathring{\mathcal{D}}(\mathring{\boldsymbol{r}})^{kl}{}_{j)} = 0.$$

Remark 13. The above observation is pertinent also to the case of non-compact $\mathcal{S}$. In particular, it suggests that the time-symmetric initial data set for the Schwarzschild spacetime, with metric (in *isotropic coordinates*)

$$\mathring{\boldsymbol{h}} = \left(1 + \frac{m}{2r}\right)^4 \boldsymbol{\delta},$$

is potentially unsuitable (as background initial data) for the application of the Friedrich–Butscher method as $\mathring{\boldsymbol{h}}$ is not an Einstein metric.

[2]In fact, it is straightforward to check that in dimension 3 the harmonic curvature condition is equivalent to local conformal flatness and constant scalar curvature.

We will see in Section 4.3.3 —see Proposition 8— that non-existence of tracefree Codazzi tensors is, in a sense, stable under perturbations of the metric. This will be used in the sufficiency argument of the same section —see Proposition 9.

Proposition 2. Let $\boldsymbol{h}$ be a Riemannian metric of non-negative sectional curvature. If η_{ij} is a Codazzi tensor of constant trace, then it is covariantly-constant —i.e. $D_k\eta_{ij} = 0$. Moreover, if the sectional curvatures are not vanishing everywhere, then η_{ij} is a constant multiple of h_{ij}.

The proof follows by establishing positivity of the integrand in the following identity

$$\int_{\mathcal{S}} \left(\|D\boldsymbol{Y}\|^2 + r_{ij}Y_k{}^iY^{kj} - r_{ikjl}Y^{ij}Y^{kl} \right) \, d\mu \tag{4.1.17}$$

—see [42] for full details.

4.2 Conformally-rigid hyperbolic initial data

From the previous two sections, we know that the existence of either a non-trivial conformal Killing vector or a non-trivial tracefree Codazzi tensor is undesirable for the application of the Friedrich–Butscher method on compact manifolds. Moreover, it was noted earlier that a Riemannian manifold of negative-definite Ricci curvature cannot admit a globally-defined conformal Killing field, rendering such a manifold a natural first candidate for the background manifold $(\mathcal{S}, \mathring{\boldsymbol{h}})$.

Due to the highly-coupled nature of the auxiliary system of equations, $\Psi = 0$, the tractability of the required analysis is, of course, dependent on the specific properties of the background manifold, $(\mathcal{S}, \mathring{\boldsymbol{h}})$. In particular, if we consider a manifold $(\mathcal{S}, \mathring{\boldsymbol{h}})$ that is Einstein (or, equivalently, a *space form* since we are in dimension 3):

$$\mathring{r}_{ij} = \tfrac{1}{3}\mathring{r}\mathring{h}_{ij},$$

with $\mathring{r}$ (necessarily) constant, then $D_v\widetilde{\Psi}$ simplifies significantly. The requirement that $\mathring{r}_{ij}$ be negative-definite is then simply that $\mathring{r}$ be negative.

Moreover, we would also like to exclude the possibility of a non-trivial tracefree Codazzi tensor —i.e. to ensure that $\ker(\mathring{\mathcal{D}}) \cap \mathscr{S}_0^2(\mathcal{S}; \mathring{\boldsymbol{h}}) = \{0\}$. Now, in the case of hyperbolic manifolds —see [57] and also also [58]— the space of tracefree Codazzi tensors coincides with the space of *essential conformally flat deformations* —i.e. one has

$$\ker\{\mathring{\mathcal{D}} : \mathscr{S}_0^2(\mathcal{S}; \mathring{\boldsymbol{h}}) \to \mathcal{J}(\mathcal{S})\} \simeq \ker\, H/\mathring{L}(\Lambda^1(\mathcal{S})),$$

where H denotes the *linearised Cotton map* —see Section 4.4 for more details. Hence, in the search for a suitable background metric, we are naturally led to the notion of *conformal rigidity* —see Definition 11— the requirement of which places additional restrictions on the topology of $\mathcal{S}$.

4.2.1 Closed hyperbolic 3-manifolds

Hyperbolic manifolds remain an active area of research in geometry and topology, due to their importance in the decomposition of 3−manifolds according to the Geometrisation Conjecture.

Definition 10. A *hyperbolic* 3−*manifold*, $(\mathcal{S}, \boldsymbol{h})$, is a manifold of dimension 3 equipped with a hyperbolic Riemannian metric —i.e. an Einstein metric of (constant) negative scalar curvature. [3]

[3]In dimensions $n > 3$, the term *hyperbolic* is reserved for manifold of constant negative *sectional* curvature, a stronger condition than that of being Einstein. In dimension 3, the two conditions are equivalent.

In this chapter, we will be concerned with *closed* hyperbolic 3−manifolds. We recall here for context some relevant theorems of Riemannian geometry, the proofs which are far beyond the scope of this thesis —we refer the interested reader to [59] for more details. Recall that the Killing–Hopf Theorem implies that a *complete, connected* hyperbolic manifold is necessarily isometric to a quotient of the hyperbolic space, $\mathbb{H}^n$, by a discrete group of its isometries —i.e. a *Kleinian group*. Moreover, in the case of compact manifolds, completeness is guaranteed automatically by the Hopf–Rinow Theorem, and hence the manifolds of interest here possess a metric which is *locally* of the following form

$$ds^2 = \frac{4(dx^2 + dy^2 + dz^2)}{(1 - (x^2 + y^2 + z^2))^2}.$$

—i.e. locally isometric to the hyperbolic plane. While the requirement of positive curvature imposes strict topological restrictions on $\mathcal{S}$ —see [42]— the requirement of negative curvature is much less restrictive. One procedure for the construction of hyperbolic 3−manifolds is that of Dehn surgery: one removes a link, $\mathscr{L}$, from $\mathbb{S}^3$ and then glues in 2−tori (one for each connected component of the link) by identification of their boundaries with $\partial(\mathbb{S}^3 \setminus \mathscr{L})$ —the ways in which the boundaries may be identified are parametrised by a set of *slopes*, one for each connected component of the link. A theorem of Lickorish–Wallace (see [59]) guarantees that every 3−manifold topology may be obtained via Dehn surgery on some link in $\mathbb{S}^3$. A theorem of Thurston (see Section 6.26 [42]) establishes then that, if $\mathbb{S}^3 \setminus \mathscr{L}$ admits a complete hyperbolic metric, so too do the Dehn-surgered manifolds, for all but finitely many choices of slopes. Moreover, if $\mathscr{L}$ is a knot, another theorem of Thurston establishes that the manifold $\mathbb{S}^3 \setminus \mathscr{L}$ admits a complete hyperbolic metric if and only if $\mathscr{L}$ is not a *torus knot* or a *satellite knot*. When it exists, the hyperbolic metric on the Dehn-surgered (or *Dehn-filled*) manifold is unique by virtue of the Mostow Rigidity Theorem.

For what follows, we will be interested in the manifolds resulting from Dehn surgery on knots since the existence of Codazzi tensors on such manifolds has been addressed in [60] —see the next section.

Remark 14. For the purposes of Chapter 7, we note here that the manifolds resulting from Dehn surgery on a knot have vanishing first Betti number ($b_1 = 0$). Intuitively, this can be understood from the fact that all holes are filled when the 2−torus is glued in; the resulting cohomology contains only torsion groups, arising from the twisting of the 2−torus boundary, as determined by the slopes prescribed in the gluing procedure. This may be demonstrated more rigorously using the *Mayer–Vietoris sequence*. The fact that $b_1 = 0$ will be important when we study the full CCEs in Chapter 7 since, by Hodge's Theorem (see [61], for example), the Betti number is equal to the nullity of the Hodge Laplacian on 1−forms, which we will need to be an isomorphism for application of the IFT.

4.2.2 Conformal rigidity and the (non-)existence of Codazzi tensors

Following from the discussion in Section 4.1.4, a closed hyperbolic manifold does not admit any global conformal Killing vector fields, since the Ricci tensor in this case is negative-definite. It remains to investigate the existence of tracefree Codazzi tensors; we will see that the question of the existence of tracefree Codazzi tensors on a hyperbolic manifold is connected to its conformal properties under metric perturbations. More precisely, the space of tracefree Codazzi tensors on a hyperbolic manifold is precisely the space of *essential conformally flat deformations* —by *essential*, we mean L^2−orthogonal to metric perturbations of the form $\mathring{L}(\boldsymbol{X})$ (see Remark 15).

Consider first the linearisation of the Cotton–York tensor, $\mathcal{H}[\boldsymbol{h}]_{ij}$, about a background metric $\mathring{\boldsymbol{h}}$:

$$
\begin{aligned}
H(\boldsymbol{\eta})_{ij} &\equiv \frac{d}{d\tau}\mathcal{H}(\mathring{\boldsymbol{h}}+\tau\boldsymbol{\eta})_{ij}\Big|_{\tau=0} \\
&= \mathring{\epsilon}^{kl}{}_{(i}(\mathring{D}_{|k}D\mathrm{Ric}(\boldsymbol{\eta})_{l|j)} - C(\boldsymbol{\eta})^{m}{}_{|k|j)}\mathring{r}_{lm}) + \eta_{(i}{}^{k}\mathring{\mathcal{H}}_{j)k} - \tfrac{1}{2}\eta\mathring{\mathcal{H}}_{ij}
\end{aligned}
$$

with indices raised using $\mathring{h}^{ij}$. Here, $\eta \equiv \mathrm{tr}_{\mathring{\boldsymbol{h}}}(\boldsymbol{\eta})$, the operator $C(\cdot)^{i}{}_{jk}$ is the linearisation of the Christoffel symbols, as in (4.1.7), and $D\mathrm{Ric}(\boldsymbol{\eta})_{ij}$ is the linearised Ricci operator acting on the metric perturbation η_{ij} —see equation (4.1.6).

Definition 11. Following [62], a conformally flat manifold $(\mathcal{S},\boldsymbol{h})$ will be said to be *conformally-rigid* if the space of essential conformally flat deformations is trivial —i.e. if

$$\ker\, H/\mathring{L}(\Lambda^1(\mathcal{S})) = \{\mathbf{0}\},$$

where H denotes the *linearised Cotton map* —see Section 4.4.

Remark 15. Note that, indeed, $\mathrm{Im}\ \mathring{L} \subseteq \ker\ H$, as a consequence of the fact that the Cotton–York tensor is conformally-covariant and a metric perturbation in $\mathrm{Im}\ \mathring{L}$ corresponds to an infinitesimal conformal diffeomorphism —see Section 4.4.1 for more details. The space $\ker\ H/\mathring{L}(\Lambda^1(\mathcal{S}))$ is sometimes called the *premoduli space of conformally-flat structures* around $[\mathring{\boldsymbol{h}}]$. By the "Splitting Lemma", Lemma 3,

$$\ker\ H/\mathring{L}(\Lambda^1(\mathcal{S})) \simeq \ker\ H \cap \ker\ \mathring{\delta}.$$

The connection between the notion of conformal-rigidity and the (non-)existence of tracefree Codazzi tensors is made precise by the following Proposition from [58], the proof of which given in Appendix A.1.

Proposition 3. Let $(\mathcal{S},\mathring{\boldsymbol{h}})$ be a closed hyperbolic manifold, then

$$\ker\{\mathring{\mathcal{D}} : \mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}}) \to \mathcal{J}(\mathcal{S})\} = \ker\ H \cap \ker\ \mathring{\delta}.$$

Hence, if $\mathring{\boldsymbol{h}}$ is conformally-rigid then it admits no non-trivial tracefree Codazzi tensors.

The existence of a family of such manifolds is guaranteed by a theorem of Kapovich —see [60]— which states, roughly, that for all but finitely-many choices of slope, s, the Dehn-filled hyperbolic manifold $\mathcal{S} = M(s)$ is conformally-rigid. The proof is beyond the scope of this thesis.

The results of the previous section can be summarised as follows:

Proposition 4. Let $(\mathcal{S},\mathring{\boldsymbol{h}})$ be a closed hyperbolic, conformally rigid manifold. Then $(\mathcal{S},\mathring{\boldsymbol{h}})$ admits neither global conformal Killing vectors nor global tracefree Codazzi tensors.

4.2.3 The background initial data sets

In the following, let $(\mathcal{S},\mathring{\boldsymbol{h}})$ be a closed hyperbolic manifold with sectional curvature normalised to $k=-1$ (or, equivalently, with $\mathring{r}=-6$). Then, for any given constant $\mathring{K}$, the tensor fields

$$\mathring{h}_{ij}, \qquad \mathring{K}_{ij} = \tfrac{1}{3}\mathring{K}\mathring{h}_{ij}, \tag{4.2.1}$$

over $\mathcal{S}$ constitute a solution to the Einstein constraint equations with constant mean extrinsic curvature $\mathring{K}$ and with cosmological constant given by

$$\lambda = \tfrac{1}{3}(\mathring{K}^2 - 9),$$

as can be readily seen from the Hamiltonian constraint (1.2.1a). Initial data of this type will be called *hyperbolic initial data*. We remark in passing that it was shown in [62] that the subclass of such initial data with $\lambda = 0$ is Cauchy stable in the expanding time-direction.

Remark 16. Note that here we are choosing to normalise the intrinsic curvature, which in turn fixes the value of the cosmological constant, once the extrinsic curvature has been given. One could alternatively rescale the intrinsic and extrinsic curvatures appropriately so as to normalise the cosmological constant. The former option is chosen since, in the subsequent analysis, it is the intrinsic geometry of $(\mathcal{S}, \mathring{\boldsymbol{h}})$ that will be of primary importance.

4.3 Application of the Friedrich–Butscher method

As discussed in the previous section, the existence of a non-trivial conformal Killing vector field and/or of a tracefree Codazzi tensor obstructs the direct implementation of the Friedrich–Butscher method. Given that such obstructions are not present in the case of conformally rigid hyperbolic Riemannian manifolds and that the structure of the linearised auxiliary extended constraint map $D_v\widetilde{\Psi}$ is substantially simplified for background initial data sets that have constant mean curvature, in the remainder of this chapter we restrict our attention to background initial data of this form.

4.3.1 Statement of the main result

The (unique) solution to the extended Einstein constraint equations associated to (4.2.1) is obtained by setting $\mathring{S}_{ij} = \mathring{\bar{S}}_{ij} = 0$ —see (3.3.13a)–(3.3.13b). Note that the sign of λ is dependent on the choice of $\mathring{K}$: $\lambda < 0$ for $|\mathring{K}| < 3$, $\lambda = 0$ for $\mathring{K} = \pm 3$ and $\lambda > 0$ for $|\mathring{K}| > 3$. In the following it will prove convenient to define the constants

$$\alpha \equiv -4 + \frac{2}{9}\mathring{K}^2, \qquad \beta \equiv -4 + \frac{8}{9}\mathring{K}^2. \tag{4.3.1}$$

Define also for $s \geq 4$ the Banach spaces[4] $\mathcal{X}^s, \mathcal{Y}^s, \mathcal{Z}^s$, as follows

$$\begin{aligned}
\mathcal{X}^s &\equiv H^{s-1}(\mathscr{C}(\mathcal{S})) \times H^{s-1}(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}})) \times H^{s-1}(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}})),\\
\mathcal{Y}^s &\equiv H^{s}(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})) \times H^{s}(\Lambda^1(\mathcal{S})) \times H^{s}(\Lambda^1(\mathcal{S})) \times H^{s}(\mathscr{S}^2(\mathcal{S})),\\
\mathcal{Z}^s &\equiv H^{s-2}(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})) \times H^{s-2}(\Lambda^1(\mathcal{S})) \times H^{s-2}(\Lambda^1(\mathcal{S})) \times H^{s-2}(\mathscr{S}^2(\mathcal{S})).
\end{aligned}$$

and where the norms are defined with respect to the background metric $\mathring{\boldsymbol{h}}$ —unless explicitly indicated otherwise, all H^s-norms from now on will be defined with respect to $\mathring{\boldsymbol{h}}$.

Remark 17. That the image of $\widetilde{\Psi} : \mathcal{X}^s \times \mathcal{Y}^s$ is indeed contained in $\mathcal{Z}^s$ may be easily checked using the Schauder ring property —see Section 2.3.2.

We are now in a position to state our main theorem:

[4] As noted in Section 2.3.2, $H^l(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}}))$ is indeed a sub-Banach space of $H^l(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}}))$.

Theorem 2. Let $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$ be a conformally rigid hyperbolic initial data set with constant mean extrinsic curvature $\mathring{K}$ satisfying

$$\beta \notin Spec\big(-\mathring{\Delta} : C^\infty(\mathcal{S}) \to C^\infty(\mathcal{S})\big). \tag{4.3.2}$$

Then, there exists an open neighbourhood $\mathcal{U} \subseteq \mathcal{X}^s$ of $(\mathbf{0},\mathbf{0},\mathbf{0})$, an open neighbourhood $\mathcal{W} \subseteq \mathcal{Y}^s$ of $(\mathring{\boldsymbol{h}}, \mathbf{0}, \mathbf{0}, \mathring{\boldsymbol{K}})$ and a smooth map $\nu : \mathcal{U} \to \mathcal{W}$ such that, defining

$$u \equiv (\phi, \boldsymbol{T}, \bar{\boldsymbol{T}}), \qquad \nu(u) \equiv \big(\boldsymbol{\chi}(u), \bar{\boldsymbol{X}}(u), \boldsymbol{X}(u), \boldsymbol{h}(u)\big),$$

the following assertions hold:

i) for each $(\phi, \boldsymbol{T}, \bar{\boldsymbol{T}}) \in \mathcal{U}$,

$$w(u) \equiv \big(\boldsymbol{\chi}(u) + \tfrac{1}{3}(\phi + \mathring{K})\mathring{\boldsymbol{h}},\ \bar{\boldsymbol{S}}(\bar{\boldsymbol{X}}(u), \bar{\boldsymbol{T}}),\ \boldsymbol{S}(\boldsymbol{X}(u), \boldsymbol{T}),\ \boldsymbol{h}(u)\big)$$

is a solution to the extended constraint equations with cosmological constant $\lambda = (\mathring{K}^2 - 9)/3$;

ii) the map $u \mapsto w(u)$ is injective for $\mathring{K} \neq 0$ —that is to say that each such solution w corresponds to a unique choice of free data $u = (\phi, \boldsymbol{T}, \bar{\boldsymbol{T}})$. Moreover, it is injective for $\mathring{K} = 0$ if we restrict the free datum ϕ to the sub-Banach space $\bar{H}^{s-1}(\mathscr{C}(\mathcal{S}))$.

Remark 18. Notice that when $|\mathring{K}| \leq \sqrt{9/2}$ —and, in particular in the time symmetric case, $\mathring{K} = 0$— condition (4.3.2) is satisfied trivially since $\beta < 0$ but $-\mathring{\Delta}$ is positive-semi-definite. Note that in this case the cosmological constant is negative ($\lambda < 0$). Moreover, since the spectrum of $-\mathring{\Delta}$ is discrete, condition (4.3.2) excludes only countably-many values of $\mathring{K}$.

The theorem will be proven in two stages in the forthcoming sections, by means of Propositions 6 and 9. In Section 4.4 we describe a parametrisation of the free data through the use of the linearised Cotton–York map, based on the results of [58, 63], and summarised in Corollary 1.

4.3.2 Existence of candidate solutions

The purpose of this section is to show the existence of perturbative solutions to the auxiliary system in the case of conformally rigid hyperbolic initial data sets.

Since the background solution admits no conformal Killing vectors and no non-trivial tracefree Codazzi tensors, the operators $\mathring{L}$ and $\mathring{\mathcal{D}}$ are both injective. Therefore, by the splitting lemma, the following are isomorphisms for $s \geq 4$:

$$\begin{aligned}
&\mathring{\delta} \circ \mathring{L} : H^s(\Lambda^1(\mathcal{S})) \to H^{s-2}(\Lambda^1(\mathcal{S})),\\
&\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}} : H^s(\mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})) \to H^{s-2}(\mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})).
\end{aligned}$$

Since the background initial data, being hyperbolic, consists of an Einstein metric and umbilical extrinsic curvature, the linearisation of the auxiliary extended constraint map in the direction of the determined fields, $D_v\widetilde{\Psi}$, takes the particularly simple form

$$D_v\widetilde{\Psi} \cdot (\boldsymbol{\sigma}, \bar{\boldsymbol{\xi}}, \boldsymbol{\xi}, \boldsymbol{\gamma}; \phi, \bar{\boldsymbol{T}}, \boldsymbol{T}) = \begin{pmatrix}
\mathring{\mathcal{D}}^*\big(\mathring{\mathcal{D}}(\boldsymbol{\sigma}) - \frac{1}{3}\mathring{K}\mathring{\mathcal{D}}(\boldsymbol{\gamma}) - \mathring{\star}\mathring{L}(\bar{\boldsymbol{\xi}})\big)_{ij} \\
\mathring{\delta} \circ \mathring{L}(\bar{\boldsymbol{\xi}})_i \\
\mathring{\delta} \circ \mathring{L}(\boldsymbol{\xi})_i \\
\frac{1}{2}\mathring{\Delta}_L\gamma_{ij} - \frac{1}{2}\alpha\bar{\gamma}_{ij} - \frac{1}{6}\beta\gamma\mathring{h}_{ij} + \frac{1}{3}\mathring{K}\sigma_{ij} - \mathring{L}(\boldsymbol{\xi})_{ij}
\end{pmatrix}.$$

Remark 19. Let $(A_{ij},\ \bar{B}_i,\ B_i,\ C_{ij}) \in \mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}}) \times \Lambda^1(\mathcal{S}) \times \Lambda^1(\mathcal{S}) \times \mathscr{S}^2(\mathcal{S})$ be an arbitrary H^{s-2} section. Then in order to establish whether $D_v\widetilde{\Psi}$ is an isomorphism, we are concerned with solving the system of equations

$$\mathring{\mathcal{D}}^*(\mathring{\mathcal{D}}(\boldsymbol{\sigma}) - \tfrac{1}{3}\mathring{K}\mathring{\mathcal{D}}(\boldsymbol{\gamma}) - \mathring{\star}\mathring{L}(\bar{\boldsymbol{\xi}}))_{ij} = A_{ij}, \tag{4.3.3a}$$

$$\mathring{\delta} \circ \mathring{L}(\bar{\boldsymbol{\xi}})_i = \bar{B}_i, \tag{4.3.3b}$$

$$\mathring{\delta} \circ \mathring{L}(\boldsymbol{\xi})_i = B_i, \tag{4.3.3c}$$

$$\mathring{\Delta}_L\gamma_{ij} - \alpha\bar{\gamma}_{ij} - \tfrac{1}{3}\beta\gamma\mathring{h}_{ij} + \tfrac{2}{3}\mathring{K}\sigma_{ij} - 2\mathring{L}(\boldsymbol{\xi})_{ij} = C_{ij}, \tag{4.3.3d}$$

where here γ and $\bar{\gamma}_{ij}$ denote the trace and tracefree parts of γ_{ij} with respect to $\mathring{\boldsymbol{h}}$, and the constants $\alpha,\ \beta$ are as defined in (4.3.1). Note the semi-decoupled form of the system: one can first solve (4.3.3b)-(4.3.3c), and then proceed to solve (4.3.3a) and (4.3.3d), in turn.

In order to address injectivity of the map ν, we also need to establish injectivity of $D_u\widetilde{\Psi}$, the linearisation of $\widetilde{\Psi}$ in the direction of the free data. For a general data set $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$, we have

$$D_u\widetilde{\Psi}\cdot(\phi, \bar{\boldsymbol{T}}, \boldsymbol{T}) = \begin{pmatrix} -\frac{1}{6}\mathring{L}(d\phi)_{jk} - \frac{1}{2}\mathring{\epsilon}_{kil}\mathring{D}^l\bar{T}_j{}^i - \frac{1}{2}\mathring{\epsilon}_{jil}\mathring{D}^l\bar{T}_k{}^i \\ \mathring{\epsilon}_{ljk}\mathring{K}^{ij}T_i{}^k + \mathring{D}^i\bar{T}_{il} \\ -\mathring{\epsilon}_{ikl}\mathring{K}^{jk}\bar{T}_j{}^l + \mathring{D}^jT_{ij} \\ -T_{ij} + \frac{1}{3}(\mathring{K}_{ij} + \mathring{K}\mathring{h}_{ij})\phi \end{pmatrix}. \tag{4.3.4}$$

Remark 20. For pure-trace $\mathring{K}_{ij}$, the case of interest at present, it is clear that if the above map is to be injective then we at least require $T_{ij}, \bar{T}_{ij}$ to be tracefree with respect to $\mathring{\boldsymbol{h}}$. If not, then

$$\phi = \text{const.}, \quad T_{ij} = \tfrac{4}{3}\mathring{K}\phi\mathring{h}_{ij}, \quad \bar{T}_{ij} = \bar{T}\mathring{h}_{ij}$$

with $\bar{T} = \text{const.} \neq 0$ would comprise a non-trivial element of the kernel of $D_u\widetilde{\Psi}$. This further justifies the use of the ansatz (4.1.3b)-(4.1.3c).

In order to prove the existence of candidate solutions, we will need the following Proposition:

Proposition 5. The operator $\mathring{P}_L : H^k(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})) \to H^{k-2}(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}}))$, $k \geq 4$, is an isomorphism when $(\mathcal{S}, \mathring{\boldsymbol{h}})$ is an Einstein manifold of negative scalar curvature (i.e. a hyperbolic manifold).

Proof. Fix $\mathring{r} = -6$, without loss of generality. Suppose that $\eta_{ij} \in \mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})$ is in the kernel of $\mathring{P}_L$:

$$\mathring{P}_L\eta_{ij} \equiv (\mathring{\Delta}_L + 4)\eta_{ij} \equiv -(\mathring{\Delta} + 2)\eta_{ij} = 0. \tag{4.3.5}$$

Then, taking the divergence, and commuting derivatives,

$$\begin{aligned} 0 &= \mathring{D}^j(\mathring{\Delta}_L + 4)\eta_{ij} \\ &= \mathring{\Delta}_H(\mathring{\delta}(\boldsymbol{\eta}))_i + 4\mathring{\delta}(\boldsymbol{\eta})_i \\ &= (-\mathring{\Delta} + 2)\mathring{\delta}(\boldsymbol{\eta})_i \end{aligned}$$

where, recall that $\mathring{\Delta}_H$ is the Hodge Laplacian with respect to $\mathring{\boldsymbol{h}}$, and in the second line we are using the well-known fact that $\delta \circ \Delta_L = \Delta_H \circ \delta$ for metrics of covariantly-constant curvature. By positive-definiteness of $(-\mathring{\Delta} + 2)$ we see that $\mathring{\delta}(\boldsymbol{\eta}) = 0$.

Now, note that

$$\begin{aligned}
\mathring{D}_k \mathring{D}_{(i}\eta_{j)}{}^k &= \mathring{D}_{(i}\mathring{\delta}(\boldsymbol{\eta})_{j)} + \mathring{r}_{(i}{}^k\eta_{j)k} - \mathring{r}_{ikjl}\eta^{kl} \\
&= \mathring{D}_{(i}\mathring{\delta}(\boldsymbol{\eta})_{j)} - 3\mathring{r}_{\{i}{}^k\eta_{j\}k} + \tfrac{1}{2}\mathring{r}\eta_{ij} \\
&= \mathring{D}_{(i}\mathring{\delta}(\boldsymbol{\eta})_{j)} + 3\eta_{ij}.
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}}(\boldsymbol{\eta})_{ij} &= -\mathring{\Delta}\eta_{ij} + \tfrac{1}{2}\mathring{D}_k\mathring{D}_i\eta_j{}^k + \tfrac{1}{2}\mathring{D}_k\mathring{D}_j\eta_i{}^k - \tfrac{1}{3}(\mathring{D}^k\mathring{D}^l\eta_{kl})\mathring{h}_{ij} \\
&= -\mathring{\Delta}\eta_{ij} + \mathring{D}_{(i}\mathring{\delta}(\boldsymbol{\eta})_{j)} - \tfrac{1}{3}(\mathring{D}^k\mathring{D}^l\eta_{kl})\mathring{h}_{ij} - 3\eta_{ij} \\
&= \mathring{P}_L\eta_{ij} - \eta_{ij} \\
&= -\eta_{ij},
\end{aligned}$$

where in the third line we are using $\mathring{\delta}(\boldsymbol{\eta}) = 0$ and in the fourth we are using (4.3.5). However, clearly $\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}}$ is positive-definite, and so we find that $\eta_{ij} = 0$ —that is to say, $\mathring{P}_L$ is injective on $H^4(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}}))$. Note that $\mathring{P}_L$ is self-adjoint and so, by the Fredholm alternative, $\mathring{P}_L$ maps onto $H^2(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}}))$ provided it is injective on the latter space. To show that this is the case, we apply the above argument once more —note that we have to take third derivatives in the above argument, but by elliptic regularity ker $\mathring{P}_L \cap H^2(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}})) =$ ker $\mathring{P}_L \cap \Gamma(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}}))$, so we are indeed justified in taking third derivatives. □

The existence of solutions to the auxiliary system (i.e. the candidate solutions) is established in the following proposition:

Proposition 6. Let $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$ be a conformally rigid hyperbolic initial data set with (constant) mean extrinsic curvature $\mathring{K}$ satisfying condition (4.3.2). Then $D_v\widetilde{\Psi} : \mathcal{X}^s \to \mathcal{Z}^s$, $(s \geq 4)$ is a Banach space isomorphism, and so (by the IFT) there exist open neighbourhoods $(0, \mathbf{0}, \mathbf{0}) \in \mathcal{V} \subseteq \mathcal{Y}^s$ and $(\mathring{\boldsymbol{K}}, \mathbf{0}, \mathbf{0}, \mathring{\boldsymbol{h}}) \in \mathcal{U} \subseteq \mathcal{X}^s$ and a Fréchet differentiable map $\nu : \mathcal{U} \to \mathcal{V}$ mapping free data to solutions of the auxiliary system $\widetilde{\Psi} = 0$. Moreover the map ν is injective.

Proof.

Injectivity of $\mathbf{D_v}\widetilde{\boldsymbol{\Psi}}$. Take $A_{ij} = C_{ij} = 0$, $B_i = \bar{B}_i = 0$ in equations (4.3.3a)-(4.3.3d). Equations (4.3.3b)-(4.3.3c) imply, firstly, that $\xi_i = \bar{\xi}_i = 0$ since the background metric admits no global conformal Killing vectors. Substituting into (4.3.3a) and (4.3.3d)

$$\mathring{\mathcal{D}}^* \circ \mathcal{D}(\boldsymbol{\sigma} - \tfrac{1}{3}\mathring{K}\boldsymbol{\gamma})_{ij} = 0, \tag{4.3.6a}$$

$$\mathring{\Delta}_L\gamma_{ij} - \alpha\bar{\gamma}_{ij} - \tfrac{1}{3}\beta\gamma\mathring{h}_{ij} + \tfrac{2}{3}\mathring{K}\sigma_{ij} = 0. \tag{4.3.6b}$$

Tracing (4.3.6b) we obtain

$$-(\mathring{\Delta} + \beta)\gamma = 0.$$

By assumption $\beta \notin \mathrm{Spec}(-\mathring{\Delta})$ and therefore $\gamma = 0$. Substituting into (4.3.6a)

$$\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}}(\boldsymbol{\sigma} - \tfrac{1}{3}\mathring{K}\bar{\boldsymbol{\gamma}})_{ij} = 0. \tag{4.3.7}$$

Now, since $\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}} : \Gamma(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}})) \to \Gamma(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}}))$ is an isomorphism, $\sigma_{ij} = \frac{1}{3}\mathring{K}\bar{\gamma}_{ij}$. Substituting into

(4.3.6b) along with $\gamma = 0$ yields

$$\mathring{\Delta}_L \bar{\gamma}_{ij} + 4\bar{\gamma}_{ij} \equiv -\mathring{\Delta}\bar{\gamma}_{ij} - 2\bar{\gamma}_{ij} = 0. \tag{4.3.8}$$

By the above Proposition, we see that $\bar{\gamma}_{ij} = 0$. Collecting everything together, we have found that

$$\sigma_{ij} = \gamma_{ij} = 0, \qquad \xi_i = \bar{\xi}_i = 0,$$

—i.e. the map $D_v\widetilde{\Psi}$ is injective.

Surjectivity of $\mathbf{D_v}\widetilde{\boldsymbol{\Psi}}$. The argument for surjectivity is similar. First, since $\mathring{\delta} \circ \mathring{L} : \Gamma(\Lambda^1(\mathcal{S}) \to \Gamma(\Lambda^1(\mathcal{S}))$ is an isomorphism, equations (4.3.3b)-(4.3.3c) admit (unique) solutions $\bar{\xi}_i,\ \xi_i$, for any given $\bar{B}_i,\ B_i$. Substituting into equations (4.3.3a) and (4.3.3d) and rearranging one obtains

$$\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}}(\boldsymbol{\varsigma} - \tfrac{1}{9}\mathring{K}\gamma\mathring{\boldsymbol{h}})_{ij} = A_{ij} + \mathring{\mathcal{D}}^*(\mathring{\star}\mathring{L}(\bar{\boldsymbol{\xi}})), \tag{4.3.9a}$$

$$\mathring{\Delta}_L\gamma_{ij} + 4\bar{\gamma}_{ij} - \tfrac{1}{3}\beta\gamma\mathring{h}_{ij} + \tfrac{2}{3}\mathring{K}\varsigma_{ij} = C_{ij} + 2\mathring{L}(\boldsymbol{\xi})_{ij}, \tag{4.3.9b}$$

where, for simplicity, we have defined

$$\varsigma_{ij} \equiv \sigma_{ij} - \tfrac{1}{3}\mathring{K}\bar{\gamma}_{ij}.$$

Note that ς_{ij} is tracefree with respect to $\mathring{\boldsymbol{h}}$. Taking the trace of (4.3.9b) one obtains

$$-(\mathring{\Delta} + \beta)\gamma = C_k{}^k,$$

which admits a unique solution, since $\beta \notin \mathrm{Spec}(-\mathring{\Delta})$ implies that $-(\mathring{\Delta}+\beta)$ is invertible. Substituting into (4.3.9a) yields

$$\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}}(\boldsymbol{\varsigma})_{ij} = A_{ij} + \mathring{\mathcal{D}}^*(\mathring{\star}\mathring{L}(\bar{\boldsymbol{\xi}}))_{ij} + \tfrac{1}{9}\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}}(\gamma\mathring{h}_{ij})$$

where γ is as determined in the previous step, for which there exists a unique solution ς_{ij}, since $\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}} : \Gamma(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})) \to \Gamma(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}}))$ is an isomorphism. Finally, substituting for γ and ς_{ij} in (4.3.9b), one obtains

$$\mathring{P}_L\bar{\gamma}_{ij} \equiv \mathring{\Delta}_L\bar{\gamma}_{ij} + 4\bar{\gamma}_{ij} = C_{ij} + 2\mathring{L}(\boldsymbol{\xi})_{ij} + \tfrac{1}{3}\beta\gamma\mathring{h}_{ij} - \tfrac{2}{3}\mathring{K}\varsigma_{ij},$$

which admits a unique solution since $\mathring{P}_L$ is an isomorphism by Proposition 5.

The previous two steps conclude the proof that $D_v\widetilde{\Psi}$ is an isomorphism, and so by the IFT there exists a map ν from the freely-prescribed data to the space of solutions of the auxiliary system $\widetilde{\Psi} = 0$. It only remains to establish the injectivity of the map ν.

Injectivity of ν: This follows from the IFT provided we can show that $D_u\widetilde{\Psi}$ is injective. Since the background initial data, being hyperbolic, has umbilical extrinsic curvature, the expression (4.3.4) simplifies to

$$\mathring{L}(d\phi)_{jk} + 3\mathring{\epsilon}_{kil}\mathring{D}^l\bar{T}_j{}^i + 3\mathring{\epsilon}_{jil}\mathring{D}^l\bar{T}_k{}^i = 0, \tag{4.3.10a}$$

$$\mathring{D}^i\bar{T}_{il} = 0, \tag{4.3.10b}$$

$$\mathring{D}^jT_{ij} = 0, \tag{4.3.10c}$$

$$T_{ij} - \tfrac{4}{9}\mathring{K}\phi\mathring{h}_{ij} = 0. \tag{4.3.10d}$$

First consider the case $\mathring{K} \neq 0$: taking the trace of the algebraic equation (4.3.10d) one finds that $\phi = 0$, and so $T_{ij} = 0$. Combining (4.3.10a)–(4.3.10b) —see Remark 4— and using $\phi = 0$, one obtains

$$(\mathring{\mathcal{D}}\bar{T})_{ijk} \equiv \mathring{D}_i \bar{T}_{jk} - \mathring{D}_j \bar{T}_{ik} = 0.$$

Now, we have assumed the non-existence of non-trivial tracefree Codazzi tensors, so $\bar{T}_{ij} = 0$. Hence, in the non–time symmetric case $\mathring{K} \neq 0$, the map ν is injective.

Consider on the other hand the time-symmetric case $\mathring{K} = 0$. Clearly, the kernel of the system contains triples of the form

$$(T_{ij}, \bar{T}_{ij}, \phi) = (\mathbf{0},\ \mathbf{0},\ \text{const.}). \tag{4.3.11}$$

We show that these are indeed the only solutions. First, note that condition (4.3.10d) (setting $\mathring{K} = 0$) again implies $T_{ij} = 0$. Now, taking the divergence of (4.3.10a), one has that

$$\begin{aligned}
0 &= \mathring{\delta}\mathring{L}(d\phi)_k + 3\mathring{\epsilon}_{kil}\mathring{D}^j \mathring{D}^l \bar{T}_j{}^i + 3\mathring{\epsilon}_{jil}\mathring{D}^j \mathring{D}^l \bar{T}_k{}^i \\
&= \mathring{\delta}\mathring{L}(d\phi)_k + \tfrac{3}{2}\mathring{\epsilon}^{jlm}\bar{T}_k{}^i \mathring{r}_{ijlm} - \tfrac{3}{2}\mathring{\epsilon}_i{}^{lm}\bar{T}^{ij}\mathring{r}_{kjlm} + 3\mathring{\epsilon}_{kjl}\mathring{D}_i \mathring{D}^l \bar{T}^{ij} \\
&= \mathring{\delta}\mathring{L}(d\phi)_k + 6\mathring{\epsilon}_{kjl}\bar{T}^{ij}\mathring{r}_i{}^l + 3\mathring{\epsilon}_{kjl}\mathring{D}^l \mathring{D}_i \bar{T}^{ij} \\
&= \mathring{\delta}\mathring{L}(d\phi)_k,
\end{aligned}$$

after commuting covariant derivatives and where in the last step we are using the fact that the background metric is Einstein, along with the fact that $\bar{T}_{ij}$ is divergence-free. Integrating by parts, one then finds that $\mathring{L}(d\phi) = 0$ —that is to say, $d\phi$ is a conformal Killing vector. Since $\mathring{\boldsymbol{h}}$ admits no non-trivial conformal Killing vectors, $d\phi = 0$ and so ϕ is constant. Proceeding as in the $\mathring{K} \neq 0$ case, we again see that $\bar{T}_{ij} = 0$, as a consequence of there being no non-trivial tracefree Codazzi tensors. By restricting the choice of ϕ to the sub-Banach space $\bar{H}^{s-1}(\mathscr{C}(\mathcal{S}))$, we clearly exclude from the kernel triples of the form (4.3.11), thereby ensuring that ν is injective. □

Remark 21. Recall the notion of *total mean extrinsic curvature*

$$\int_{\mathcal{S}} \operatorname{tr}_{\mathring{\boldsymbol{h}}}(\boldsymbol{K})\ d\mathring{\mu},$$

given here with respect to the background metric $\mathring{\boldsymbol{h}}$. The additional requirement that $\phi \in \bar{H}^{s-1}$ ensures that the resulting solutions furnished by Theorem 2 have the same total mean extrinsic curvature with respect to $\mathring{\boldsymbol{h}}$ as the background solution. While the proof guarantees a solution for any choice of sufficiently small ϕ, the injectivity of the map ν is only guaranteed if we further restrict to those ϕ in $\bar{H}^{s-1}$.

Remark 22. Note that, since the symmetric tensor $\boldsymbol{h}$ so-constructed is close to $\mathring{\boldsymbol{h}}$ in H^2, it is in C^0 and, provided we choose the perturbations to be sufficiently small, $\|\boldsymbol{h} - \mathring{\boldsymbol{h}}\|_\infty$ will be small enough so as to guarantee that $\boldsymbol{h}$ is positive-definite at each point $p \in \mathcal{S}$ and therefore is indeed a Riemannian metric.

4.3.3 Sufficiency of the auxiliary system

In this section we establish *sufficiency* of the auxiliary constraint system —that is, we show that the solutions of the auxiliary system established in the previous section are indeed solutions of the extended constraint equations.

Our argument for sufficiency is based on the integral identities (4.1.12) and (4.1.16). First, we note that

$$\int_{\mathcal{S}} \|D\boldsymbol{F}\|^2_{\boldsymbol{h}} \, d\mu_{\boldsymbol{h}} = \int_{\mathcal{S}} \left(\|\mathcal{R}(\boldsymbol{F})\|^2_{\boldsymbol{h}} + \tfrac{3}{2}\|\delta(\boldsymbol{F})\|^2 - r_{jm}F_i{}^m F^{ij} + r_{imjn}F^{ij}F^{mn} \right) d\mu_{\boldsymbol{h}} \tag{4.3.12}$$

Note that $\mathcal{R}(\boldsymbol{F})_{ij} = \delta(\boldsymbol{F})_i = 0$ if and only if $\mathcal{D}(\boldsymbol{F})_{ijk} = 0$ —that is to say, if and only if F_{ij} is a tracefree Codazzi tensor. Then, substituting equation (4.3.12) into the identity (4.1.16), one obtains

$$0 = \int_{\mathcal{S}} \Big(\|\mathcal{R}(\boldsymbol{F})\|^2_{\boldsymbol{h}} + \tfrac{3}{2}\|\delta(\boldsymbol{F})\|^2 + 2\|D\boldsymbol{A}\|^2_{\boldsymbol{h}} + \tilde{\mathscr{R}}_{\boldsymbol{h}}(\boldsymbol{A},\boldsymbol{F}) \Big) \, d\mu_{\boldsymbol{h}}, \tag{4.3.13}$$

where now

$$\tilde{\mathscr{R}}_{\boldsymbol{h}}(\boldsymbol{A},\boldsymbol{F}) \equiv -r_{ij}A^iA^j + 3\epsilon_{imn}r_j{}^n F^{jm}A^i - \tfrac{1}{2}\epsilon_j{}^{np}r_{imnp}F^{jm}A^i.$$

The important observation is that, when evaluated at $\boldsymbol{h} = \mathring{\boldsymbol{h}}$, we have

$$\tilde{\mathscr{R}}_{\mathring{\boldsymbol{h}}}(\boldsymbol{A},\boldsymbol{F}) = 2\|\boldsymbol{A}\|^2_{\mathring{\boldsymbol{h}}} \geq 0,$$

since $\mathring{r}_{ij} = -2\mathring{h}_{ij}$. Collecting together the above observations, we deduce the following:

Proposition 7. The system of equations (4.1.10a), (4.1.11a) with $\boldsymbol{h} = \mathring{\boldsymbol{h}}$, namely

$$\mathring{\mathcal{D}}^*(J)_{ij} = 0,$$
$$\mathring{\epsilon}^{ijk}\mathring{D}_iJ_{jkl} = 0,$$

admits no non-trivial solutions J_{ijk}.

Proof. Evaluating (4.3.13) at $\boldsymbol{h} = \mathring{\boldsymbol{h}}$, we have that

$$0 = \int_{\mathcal{S}} \Big(\|\mathring{\mathcal{R}}(\boldsymbol{F})\|^2_{\mathring{\boldsymbol{h}}} + \tfrac{3}{2}\|\mathring{\delta}(\boldsymbol{F})\|^2_{\mathring{\boldsymbol{h}}} + 2\|\mathring{D}\boldsymbol{A}\|^2_{\mathring{\boldsymbol{h}}} + 2\|\boldsymbol{A}\|^2_{\mathring{\boldsymbol{h}}} \Big) \, d\mathring{\mu},$$

where A_i, F_{ij} denote the decomposition (2.1.9) of J_{ijk} with respect to $\mathring{\boldsymbol{h}}$ and $d\mathring{\mu} = d\mu_{\mathring{\boldsymbol{h}}}$. Therefore $A_i = 0$ and $\mathring{\mathcal{R}}(\boldsymbol{F}) = \mathring{\delta}(\boldsymbol{F}) = 0$, the latter implying that $F_{ij} = 0$ since the background metric admits no tracefree Codazzi tensors. Hence, we see that $J_{ijk} = 0$. □

In the following, it will prove convenient to first define the operator

$$\mathcal{K}_{\boldsymbol{h}} : \mathcal{J}(\mathcal{S}) \longrightarrow \mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}}) \oplus \Lambda^1(\mathcal{S})$$

acting as

$$\mathcal{K}_{\boldsymbol{h}}(\boldsymbol{J}) = \begin{pmatrix} \mathring{\mathcal{D}}^*(\boldsymbol{J})_{ij} \\ \epsilon^{ijk}D_iJ_{jkl} \end{pmatrix}.$$

Note that Proposition 7 simply asserts the injectivity of $\mathring{\mathcal{K}} \equiv \mathcal{K}_{\mathring{\boldsymbol{h}}}$. In order to extend the argument to $\mathcal{K}_{\boldsymbol{h}}$, for $\boldsymbol{h}$ sufficiently close to $\mathring{\boldsymbol{h}}$, we will first show that the operator $\mathcal{K}_{\boldsymbol{h}}$ is elliptic and then appeal to a particular stability property of the kernel of elliptic operators. Let us first establish ellipticity:

Lemma 6. For each $\boldsymbol{h}$, the operator $\mathcal{K}_{\boldsymbol{h}}$ is first-order elliptic.

Proof. Recall from Lemma 1 that $\mathcal{J}(\mathcal{S})$ and $\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}}) \oplus \Lambda^1(\mathcal{S})$ are isomorphic as vector spaces. Therefore, in order to establish ellipticity it suffices to show that $\mathcal{K}_{\boldsymbol{h}}$ is overdetermined elliptic. Note also that the second component of $\mathcal{K}_{\boldsymbol{h}} = 0$ is equivalent to

$$D_{[i}J_{jk]l} = 0.$$

Note that a change of connection $D_i \to \mathring{D}_i$ only introduces lower-order (i.e. algebraic) terms involving J_{ijk}, so in order to show ellipticity it suffices to consider the operator $\mathring{\mathcal{K}}$, or equivalently an operator with principal part

$$\begin{pmatrix} \mathring{\mathcal{D}}^*(\boldsymbol{J})_{ij} \\ \mathring{D}_{[i}J_{jk]l}. \end{pmatrix}.$$

Accordingly, suppose $J_{ijk} \in \mathcal{J}(\mathcal{S})$ is in the kernel of the symbol map, $\sigma_\xi[\mathring{\mathcal{K}}]$, for a given fixed ξ_i, so that

$$\xi^k J_{ikj} + \xi^k J_{jki} - \tfrac{2}{3}\xi^k J_{lk}{}^l = 0, \tag{4.3.14a}$$

$$\xi_i J_{jkl} + \xi_j J_{kil} + \xi_k J_{ijl} = 0. \tag{4.3.14b}$$

Here contractions are performed with respect to $\mathring{\boldsymbol{h}}$. Contracting indices i, l in equation (4.3.14b), we obtain

$$\xi^l J_{jkl} = -\xi_j J_{kl}{}^l + \xi_k J_{jl}{}^l. \tag{4.3.15}$$

On the other hand, contracting (4.3.14a) with ξ^j, we obtain

$$\begin{aligned} 0 &= \xi^k\xi^j J_{ikj} + \xi^k\xi^j J_{jki} - \tfrac{2}{3}\xi^k\xi_i J_{lk}{}^l \\ &= \xi^k\xi^j J_{ikj} - \tfrac{2}{3}\xi^k\xi_i J_{lk}{}^l \\ &= \tfrac{1}{3}\xi_i\xi^k J_{kl}{}^l + |\boldsymbol{\xi}|^2 J_{il}{}^l, \end{aligned} \tag{4.3.16}$$

where the second line follows from the fact that $J_{ijk} = -J_{jik}$ and the third line follows from substituting (4.3.15). Contracting (4.3.16) with ξ^i, we find that $\xi^i J_{il}{}^l = 0$, which when substituted back into (4.3.16) yields $J_{il}{}^l = 0$. Substituting the latter into (4.3.14a) we see that

$$\xi^k J_{ikj} + \xi^k J_{jki} = 0. \tag{4.3.17}$$

Moreover, substitution of $J_{il}{}^l = 0$ into (4.3.15) yields

$$\xi^k J_{ijk} = 0. \tag{4.3.18}$$

Now, contracting the cyclic identity $J_{ijk} + J_{jki} + J_{kij} = 0$ with ξ^k one finds that

$$\begin{aligned} 0 &= \xi^k J_{ijk} + \xi^k J_{jki} + \xi^k J_{kij} \\ &= \xi^k J_{jki} - \xi^k J_{ikj}, \end{aligned} \tag{4.3.19}$$

where to pass from the first to the second line we have used (4.3.18) and that $J_{kij} = -J_{ikj}$. Combining equations (4.3.17) and (4.3.19) one thus concludes that

$$\xi^k J_{ikj} = 0. \tag{4.3.20}$$

Finally, contracting (4.3.14b) with ξ^i, we obtain

$$0 = |\boldsymbol{\xi}|^2 J_{jkl} + \xi_j \xi^i J_{kil} + \xi_k \xi^i J_{ijl} = |\boldsymbol{\xi}|^2 J_{jkl},$$

where the second equality follows from $\xi^k J_{ikj} = \xi^k J_{kij} = 0$. Hence, for $\xi_i \neq 0$, we see that the symbol map is injective —that is to say, $\mathcal{K}_{\boldsymbol{h}}$ is overdetermined elliptic and hence, from our earlier observation, in fact determined elliptic. □

In order to establish injectivity of $\mathcal{K}_{\boldsymbol{h}}$ we will make use of an elliptic estimate. Rather than working directly with the first-order operator $\mathcal{K}_{\boldsymbol{h}}$ we choose instead to work with the elliptic operator $\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}}$ —where we compute the L^2-adjoint with respect to the background metric, $\mathring{\boldsymbol{h}}$— to which the more standard results of second-order elliptic operators may be applied. Note that the kernel of $\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}}$ agrees with the kernel of $\mathcal{K}_{\boldsymbol{h}}$, so it suffices to show injectivity of the second-order operator. Our starting point is the following elliptic estimate for $\mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}}$: there exists $C > 0$ such that, for all $\boldsymbol{\eta} \in H^2(\mathcal{J}(\mathcal{S}))$

$$\|\boldsymbol{\eta}\|_{H^2} \leq C \left(\|\mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}}(\boldsymbol{\eta})\|_{L^2} + \|\boldsymbol{\eta}\|_{H^1} \right) \tag{4.3.21}$$

—see Theorem 1, given in Section 2.3.2. In fact, we will require a *uniform* version of the above elliptic estimate which allows for small perturbations of the metric:

Lemma 7. There exists $\varepsilon > 0$ such that, for all $\boldsymbol{h}$ satisfying $\|\boldsymbol{h} - \mathring{\boldsymbol{h}}\|_{H^2} < \varepsilon$, we have the estimate

$$\|\boldsymbol{\eta}\|_{H^2} \leq 2C \left(\|\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}}(\boldsymbol{\eta})\|_{L^2} + \|\boldsymbol{\eta}\|_{H^1} \right) \tag{4.3.22}$$

for all $\boldsymbol{\eta} \in H^2(\mathcal{J}(\mathcal{S}))$, with C as in (4.3.21), depending only on $\mathring{\boldsymbol{h}}$.

Proof. We first note that there exists some constant $\tilde{C}$ such that for any given $\boldsymbol{\eta} \in \mathcal{J}(\mathcal{S})$, we have

$$\|(\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}} - \mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}})\boldsymbol{\eta}\|_{L^2} \leq \tilde{C} \|\boldsymbol{h} - \mathring{\boldsymbol{h}}\|_{H^2} \|\boldsymbol{\eta}\|_{H^2} \tag{4.3.23}$$

—this follows from the fact that, schematically,

$$(\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}} - \mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}})\boldsymbol{\eta} \sim (\boldsymbol{h} - \mathring{\boldsymbol{h}})\partial\partial\boldsymbol{\eta} + \boldsymbol{S} \cdot \partial\boldsymbol{\eta} + (\partial\boldsymbol{S} + \boldsymbol{S} \cdot \boldsymbol{S})\boldsymbol{\eta},$$

with $\boldsymbol{S}$ denoting the transition tensor between the covariant derivatives associated to the metrics $\mathring{\boldsymbol{h}}$ and $\boldsymbol{h}$, from which it is clear then that $(\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}} - \mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}})\boldsymbol{\eta}$ may be bounded above by $\|\boldsymbol{h} - \mathring{\boldsymbol{h}}\|_{H^2} \|\boldsymbol{\eta}\|_{H^2}$.

Now, using inequality (4.3.23) we find that for all $\boldsymbol{h}$ satisfying $\|\boldsymbol{h} - \mathring{\boldsymbol{h}}\|_{H^2} < \varepsilon$, and for all $\boldsymbol{\eta} \in \mathcal{J}(\mathcal{S})$,

$$\begin{aligned}
\|\boldsymbol{\eta}\|_{H^2} &\leq C \left(\|\mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}}(\boldsymbol{\eta})\|_{L^2} + \|\boldsymbol{\eta}\|_{H^1} \right) \\
&\leq C \left(\|\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}}(\boldsymbol{\eta})\|_{L^2} + \|(\mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}} - \mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}})\boldsymbol{\eta}\|_{L^2} + \|\boldsymbol{\eta}\|_{H^1} \right) \\
&\leq C \left(\|\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}}(\boldsymbol{\eta})\|_{L^2} + \varepsilon\tilde{C}\|\boldsymbol{\eta}\|_{H^2} + \|\boldsymbol{\eta}\|_{H^1} \right),
\end{aligned}$$

with C depending only on $\mathring{\boldsymbol{h}}$. Thus, taking $\varepsilon = 1/(2C\tilde{C})$ and rearranging we have that

$$\|\boldsymbol{\eta}\|_{H^2} \leq 2C \left(\|\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}}(\boldsymbol{\eta})\|_{L^2} + \|\boldsymbol{\eta}\|_{H^1} \right) \tag{4.3.24}$$

for all $\boldsymbol{\eta} \in H^2(\mathcal{J}(\mathcal{S}))$ and for all $\|\boldsymbol{h} - \mathring{\boldsymbol{h}}\|_{H^2} < \varepsilon$ as required. □

Remark 23. The content of inequality (4.3.23) may be summarised by the statement that the map

$$\begin{array}{rccc} M: & H^2(\mathscr{S}^2(\mathcal{S})) & \longrightarrow & B(H^2(\mathcal{J}(\mathcal{S})), L^2(\mathcal{J}(\mathcal{S})) \\ & \boldsymbol{h} & \longmapsto & \mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}} \end{array}$$

is Lipschitz continuous at $\boldsymbol{h} = \mathring{\boldsymbol{h}}$ —here, $B(\cdot,\cdot)$ denotes the Banach space of bounded linear maps between the indicated Banach spaces, endowed with the operator norm— with $\tilde{C}$ the Lipschitz constant, which depends on the precise structure of $\mathcal{K}^* \circ \mathcal{K}$ and may be computed explicitly.

Assume now that the procedure described in Section 4.3.2 has been carried out —that is to say, we have established the existence of a neighbourhood of solutions to the auxiliary system. For each such solution, the corresponding zero quantities $Q_i,\ J_{ijk}$ necessarily satisfy

$$\mathcal{K}_{\boldsymbol{h}}(\boldsymbol{J}) = 0, \tag{4.3.25a}$$

$$D^i(\mathcal{L}_Q\boldsymbol{h})_{ij} - \tfrac{1}{2}D_j(\mathcal{L}_Q\boldsymbol{h})_i{}^i = 2K_{ik}J_j{}^{ik} - 2K_{jk}J^{ik}{}_i - 2KJ_j{}^i{}_i, \tag{4.3.25b}$$

the first equation collects together (4.1.10a) and (4.1.11a), while the latter is the remaining integrability condition — see Section 4.1.3. We regard the above as equations for a pair of tensor fields $\boldsymbol{Q} \in \Lambda^1(\mathcal{S}),\ \boldsymbol{J} \in \mathcal{J}(\mathcal{S})$, which we aim to prove are necessarily vanishing —at this point we forget about the definitions of the zero quantities $Q_i,\ J_{ijk}$ in terms of the unknown tensor fields.

We first use the results of the previous section to show that injectivity of the operator $\mathcal{K}_{\boldsymbol{h}}$ is stable under H^2-perturbations of the metric.

Proposition 8. Given injective $\mathring{\mathcal{K}}$, there exists $\varepsilon > 0$ such that for all metrics $\boldsymbol{h}$ satisfying $\|\boldsymbol{h} - \mathring{\boldsymbol{h}}\|_{H^2} < \varepsilon$, then there exists a constant $C > 0$, depending only on $\mathring{\boldsymbol{h}}$, such that the following estimate holds

$$\|\boldsymbol{\eta}\|_{H^2} \leq C\|\mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}}(\boldsymbol{\eta})\|_{L^2} \tag{4.3.26}$$

for all $\boldsymbol{\eta} \in H^2(\mathcal{J}(\mathcal{S}))$. For any such metric $\boldsymbol{h}$, the corresponding operator $\mathcal{K}_{\boldsymbol{h}} : \mathcal{J}(\mathcal{S}) \to \mathcal{J}(\mathcal{S})$ is injective on H^2.

Proof. Suppose not, then there exists a *failure sequence* $\{(\boldsymbol{h}^{(n)},\ \boldsymbol{\eta}^{(n)})\}$, $n \in \mathbb{N}$ —i.e. a sequence of Riemannian metrics $\boldsymbol{h}^{(n)}$ converging to $\mathring{\boldsymbol{h}}$ in H^2 and corresponding non-zero Jacobi tensors $\boldsymbol{\eta}^{(n)} \in \mathcal{J}(\mathcal{S})$ for which

$$\mathcal{K}_{(n)}(\boldsymbol{\eta}^{(n)}) = 0$$

for each $n \in \mathbb{N}$ —here, $\mathcal{K}_{(n)} \equiv \mathcal{K}_{\boldsymbol{h}^{(n)}}$. Since $\mathcal{K}_{(n)}$ is linear, we may take each $\boldsymbol{\eta}^{(n)}$ to be of unit H^2-norm. Hence, by the Rellich–Kondrakov Theorem (see 2.3), since the sequence $\{\boldsymbol{\eta}^{(n)}\}$ is bounded in H^2, there is a subsequence that is Cauchy in H^1 —let us assume without loss of generality that $\{\boldsymbol{\eta}^{(n)}\}$ is Cauchy— converging to some limit $\boldsymbol{\eta}^\bullet \in \mathcal{J}(\mathcal{S})$. We now aim to show using the inequality (4.3.24) that the sequence is in fact Cauchy in H^2. Let us restrict to a the tail of the subsequence (relabelling, if necessary) for which $\|\boldsymbol{h}^{(n)} - \mathring{\boldsymbol{h}}\| < \varepsilon$ with ε as given in Proposition 7. Applying the inequality (4.3.24) to $\boldsymbol{\eta}^{(m,n)} \equiv \bar{\boldsymbol{\eta}}^{(n)} - \bar{\boldsymbol{\eta}}^{(m)}$, with $\boldsymbol{h} = \boldsymbol{h}^{(n)}$, we have

$$
\begin{aligned}
&\|\boldsymbol{\eta}^{(m,n)}\|_{H^2} \\
&\quad \leq 2C\left(\|\mathcal{K}^*_{(n)} \circ \mathcal{K}_{(n)}(\boldsymbol{\eta}^{(m,n)})\|_{L^2} + \|\boldsymbol{\eta}^{(m,n)}\|_{H^1}\right) \\
&\quad = 2C\left(\|\mathcal{K}^*_{(n)} \circ \mathcal{K}_{(n)}(\boldsymbol{\eta}^{(m)})\|_{L^2} + \|\boldsymbol{\eta}^{(m,n)}\|_{H^1}\right) \\
&\quad = 2C\left(\|(\mathcal{K}^*_{(n)} \circ \mathcal{K}_{(n)} - \mathcal{K}^*_{(m)} \circ \mathcal{K}_{(m)})\boldsymbol{\eta}^{(m)}\|_{L^2} + \|\boldsymbol{\eta}^{(m,n)}\|_{H^1}\right)
\end{aligned} \tag{4.3.27}
$$

The second line follows from substituting for $\boldsymbol{\eta}^{(m,n)}$ in the first term and using the fact that, by assumption, $\mathcal{K}_{(n)}(\bar{\boldsymbol{\eta}}^{(n)}) = 0$; the third line follows similarly. Now,

$$
\begin{aligned}
\|(\mathcal{K}^*_{(n)} \circ \mathcal{K}_{(n)} - \mathcal{K}^*_{(m)} \circ \mathcal{K}_{(m)})\boldsymbol{\eta}^{(m)}\|_{L^2} \leq{}& \|(\mathcal{K}^*_{(n)} \circ \mathcal{K}_{(n)} - \mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}})\boldsymbol{\eta}^{(m)}\|_{L^2} \\
&+ \|(\mathcal{K}^*_{(m)} \circ \mathcal{K}_{(m)} - \mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}})\boldsymbol{\eta}^{(m)}\|_{L^2}.
\end{aligned}
$$

The right hand side goes to zero in the limit $m, n \longrightarrow \infty$, again using the Lipschitz property of M and the fact that $\boldsymbol{\eta}^{(m)}$ is bounded in H^2. Collecting together the above observations, we see from (4.3.27) that as $m, n \longrightarrow \infty$, $\boldsymbol{\eta}^{(m,n)} \longrightarrow 0$ in H^2 —i.e. the sequence $\bar{\boldsymbol{\eta}}^{(n)}$ is Cauchy in H^2, and therefore the limit $\boldsymbol{\eta}^\bullet \in \mathcal{J}(\mathcal{S})$ is in H^2. Clearly $\boldsymbol{\eta}^\bullet$ is non-zero —in fact, it is easily verified using the reverse-triangle inequality that $\|\boldsymbol{\eta}^\bullet\|_{H^2} = 1$.

Using the Lipschitz property of M once more, along with the fact that $\boldsymbol{\eta}^{(n)}$ converges to $\boldsymbol{\eta}^\bullet$ in H^2, one finds that

$$
\|\mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}}(\boldsymbol{\eta}^\bullet)\|_{L^2} = \lim_{n\to\infty} \|\mathcal{K}^*_{(n)} \circ \mathcal{K}_{(n)}(\boldsymbol{\eta}^{(n)})\|_{L^2} = 0.
$$

Hence, $\mathring{\mathcal{K}}^* \circ \mathring{\mathcal{K}}(\boldsymbol{\eta}^\bullet) = 0$, and it follows via integration by parts that $\mathring{\mathcal{K}}(\boldsymbol{\eta}^\bullet) = 0$. However, $\boldsymbol{\eta}^\bullet \in \mathcal{J}(\mathcal{S}) \setminus \{0\}$ and so we obtain a contradiction, since $\mathring{\mathcal{K}}$ is injective, as shown in Proposition 7. □

We are now in a position to prove the main result of this section:

Proposition 9. There exists an open neighbourhood $\mathcal{V}$ of $\mathring{\boldsymbol{h}} \in H^s(\mathscr{S}^2(\mathcal{S}))$, $s \geq 4$, such that for each $\boldsymbol{h} \in \mathcal{V}$, $(J_{ijk}, Q_i) = (\mathbf{0}, \mathbf{0})$ is the unique H^2 solution of (4.3.25a)–(4.3.25b).

Proof. We begin by showing that $J_{ijk} = 0$. This follows immediately from the previous proposition provided we choose $\mathcal{V}$ to be a suitably-small neighbourhood.

Having established that $J_{ijk} = 0$, (4.3.25b) implies that Q_i satisfies the integral identity (4.1.12). Hence, it follows that

$$
0 = \int_{\mathcal{S}} \left(\|D\boldsymbol{Q}\|^2_{\boldsymbol{h}} - r_{ij}Q^iQ^j\right)\, d\mu_{\boldsymbol{h}} \geq \int_{\mathcal{S}} -r_{ij}Q^iQ^j\, d\mu_{\boldsymbol{h}} \longrightarrow \int_{\mathcal{S}} 2\|\boldsymbol{Q}\|^2_{\mathring{\boldsymbol{h}}}\, d\mu_{\mathring{\boldsymbol{h}}},
$$

where convergence follows from the fact that, since $\boldsymbol{h} \to \mathring{\boldsymbol{h}}$ in H^4, we have $r[\boldsymbol{h}]_{ij} \to \mathring{r}_{ij} = -2\mathring{h}_{ij}$ in C^0 —convergence of the latter in H^2 is immediate, and an application of the Sobolev Embedding Theorem establishes convergence in C^0. Hence, provided we take $\mathcal{V}$ to be a suitably-small neighbourhood, it follows that for any $\boldsymbol{h} \in \mathcal{V}$ we necessarily have $\boldsymbol{Q} = 0$. □

Hence, it follows that for solutions $(K_{ij}, \bar{S}_{ij}, \bar{S}_{ij}, h_{ij})$ of the auxiliary system sufficiently close to the background data, the corresponding zero quantities Q_i, J_{ijk} must necessarily vanish, implying

$(K_{ij}, \bar{S}_{ij}, \bar{S}_{ij}, h_{ij})$ indeed solves the extended constraint equations. This concludes the proof of sufficiency. Collecting together Propositions 6 and 9, one obtains Theorem 2.

4.4 Parametrising the space of freely-prescribed data

We have seen that, according to Theorem 2, there exist solutions of the extended constraints corresponding to freely-prescribed data $(\phi, \boldsymbol{T}, \bar{\boldsymbol{T}})$ sufficiently close to $(\mathbf{0}, \mathbf{0}, \mathbf{0})$, where $T_{ij}, \boldsymbol{T}_{ij} \in \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})$. In this last subsection we aim to give an explicit parametrisation of the space of *smooth* freely-prescribed data, using the ideas of [58] for the construction of transverse-tracefree tensors on conformally flat manifolds —it should be emphasised that this construction is particular to conformally-flat background metrics. These ideas have previously been applied to the construction of *generalised Bowen-York data* —see [64]. We first review the basic ideas.

4.4.1 The Gasqui–Goldschmidt–Beig complex

Recall from Section 2.2 the definition of the Cotton–York tensor, which in the notation of this Chapter may be written as

$$\mathcal{H}[\boldsymbol{h}]_{ij} \equiv \mathcal{R} \circ \mathrm{Ric}[\boldsymbol{h}]_{ij}.$$

Here, we have replaced the Schouten curvature with the Ricci curvature using the fact that $\mathcal{R}(f\boldsymbol{h})_{ij} = 0$ for all (three-times differentiable) functions f. The Cotton tensor may be thought of as a third order differential operator on the metric, $\boldsymbol{h}$. Recall also that $\mathcal{H}[\boldsymbol{h}]_{ij} \in \mathscr{S}^2_0(\mathcal{S}; \boldsymbol{h})$ and that (in dimension 3) its vanishing is equivalent to (local) conformal flatness.

According to the above observations, if $\mathring{\boldsymbol{h}}$ is conformally flat, then $H(\boldsymbol{\eta}) \in \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})$ —recall from Section 4.2.2 that H denotes the linearisation of the Cotton tensor. Moreover, in the case of conformally-flat data, $H(\boldsymbol{\eta})_{ij}$ is also divergence-free since the linearisation of the third Bianchi identity $(\delta \circ \mathcal{H}[\boldsymbol{h}] = 0)$ gives

$$\begin{aligned} 0 &= \frac{d}{d\tau} \delta_{\boldsymbol{h}}(\mathcal{H}(\boldsymbol{h}))_i \bigg|_{\tau=0} \\ &= \mathring{\delta}(H(\boldsymbol{\eta}))_i - \eta^{kj} \mathring{D}_k \mathring{\mathcal{H}}_{ij} - \tfrac{1}{2} \mathring{\mathcal{H}}^{jk} \mathring{D}_i \eta_{jk} - \mathring{\mathcal{H}}_i{}^k \mathring{D}^j \eta_{jk} + \tfrac{1}{2} \mathring{\mathcal{H}}_i{}^k \mathring{D}_k \eta \\ &= \mathring{\delta}(H(\boldsymbol{\eta}))_i, \end{aligned}$$

where to pass from the second to the third line it has been used that $\mathring{\mathcal{H}}_{ij} = 0$ for a conformally flat background. Hence, $H(\boldsymbol{\eta})_{ij} \in \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})$. For completeness, we also include the following lemma, the proof of which can be found in Appendix A.1.

Lemma 8. Let $(\mathcal{S}, \mathring{\boldsymbol{h}})$ be a conformally flat manifold, then $\mathrm{Im}\ \mathring{L} \subseteq H$.

The above properties of the Cotton–York tensor are expressed succinctly in the *Gasqui–Goldschmidt–Beig* elliptic complex[5]:

$$0 \to \Gamma(\Lambda^1(\mathcal{S})) \xrightarrow{\mathring{L}} \Gamma(\mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})) \xrightarrow{H} \Gamma(\mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})) \xrightarrow{\mathring{\delta}} \Gamma(\Lambda^1(\mathcal{S})) \to 0,$$

which holds for any conformally flat manifold $(\mathcal{S}, \mathring{\boldsymbol{h}})$ —see [61] for some of the basic properties of an elliptic complex. For general interest, note that the above complex has been generalised to higher-valence tensors in the work of Andersson, Bäckdahl, and Joudioux —see [65, 66].

[5]The complex was first identified by Gasqui–Goldschmidt in [63] and was rediscovered by Beig in [58]

Theorem. Let $\mathcal{S}$ be closed, then all cohomologies are finite-dimensional and the following expressions of *Poincaré duality* hold:

(i) $\ker\ \mathring{\delta}/\mathrm{Im}\ H \simeq \ker\ H/\mathrm{Im}\ \mathring{L}$;

(ii) $\ker\ \mathring{L} \simeq \Lambda^1(\mathcal{S})/\mathrm{Im}\ \mathring{\delta}$.

For a proof see [58]. Now, conformal-rigidity can be re-expressed simply as $\ker\ H/\mathrm{Im}\ \mathring{L} = \{\mathbf{0}\}$. Hence, for our class of conformally-rigid hyperbolic initial data, it follows that the map

$$H : \Gamma(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}})) \to \Gamma(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}}))$$

is, in fact, surjective.

4.4.2 The parametrisation

The ideas of the previous section can now be used to obtain a parametrisation of (smooth) free data $T_{ij},\ \bar{T}_{ij}$:

Corollary 1. Let $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$ be as in Theorem 2, and let $\mathcal{U}$ be the neighbourhood of the free data described there. Then, given $s \geq 4$, there exists a subset

$$\tilde{\mathcal{U}} \subset \mathcal{B}_{\boldsymbol{\eta}} \equiv \Gamma(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}}))),$$

open in the H^{s+3} topology, such that:

i) for each $\boldsymbol{\eta}, \bar{\boldsymbol{\eta}} \in \tilde{\mathcal{U}}$ there exists a solution to the extended constraint equations with smooth free data

$$T_{ij} = H(\boldsymbol{\eta})_{ij}, \qquad \bar{T}_{ij} = H(\bar{\boldsymbol{\eta}})_{ij}; \tag{4.4.1}$$

ii) all smooth free data $\boldsymbol{T}, \bar{\boldsymbol{T}} \in \mathcal{U}$ may be obtained in the form (4.4.1), for some $\boldsymbol{\eta}, \bar{\boldsymbol{\eta}} \in \tilde{\mathcal{U}}$.

For a given $T_{ij}\ \bar{T}_{ij}$, the choice of $\eta_{ij}, \bar{\eta}_{ij}$ in (4.4.1) is unique up to the addition of elements in $\mathrm{Im}(\mathring{L})$.

Proof. Take $\tilde{\mathcal{U}} \equiv H^{-1}(\mathcal{U} \cap \mathrm{Im}(H))$. The map

$$H : H^{s+2}(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}})) \to H^{s-1}(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}}))$$

is continuous, so $\tilde{\mathcal{U}}$ is open in $\mathcal{B}_{\boldsymbol{\eta}}$. Applying Theorem 2 with free data (4.4.1) establishes *(i)*. By assumption of conformal rigidity (see the above discussion)

$$H : \Gamma(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}})) \to \Gamma(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}}))$$

is surjective, so $H(\tilde{\mathcal{U}}) = \mathcal{U}$, establishing *(ii)*. Uniqueness (up to addition of elements in $\mathrm{Im}\ \mathring{L}$) also follows from the assumption of conformal rigidity. □

Remark 24. The parametrisation of TT tensors described above holds more generally for any conformally flat and conformally rigid background metric, $\mathring{\boldsymbol{h}}$. However, as discussed in 4.1.4, a conformally flat but non-Einstein metric possesses a non-trivial tracefree Codazzi tensor, rendering it unsuitable as a background metric for the Friedrich–Butscher method, at least in its current form.

4.5 Concluding remarks

The results presented in this chapter demonstrate that conformally-rigid hyperbolic manifolds provide a convenient testing ground for the application of the Friedrich–Butscher method. When supplemented with an umbilical extrinsic curvature, the resulting background initial data set admits non-linear perturbative solutions to the ECEs with freely-prescribed mean extrinsic curvature, which need not be constant (unlike that of the background data set), and freely-prescribed TT parts of the electric and magnetic Weyl curvatures under the “projected” York split. Moreover, it was shown that for such background initial data the freely-prescribed data may be explicitly parametrised by means of the linearised Cotton–York map.

In seeking to generalise the analysis of this chapter to a broader class of background data sets, there are two natural directions in which to go, depending on whether we wish to consider a more complicated intrinsic or extrinsic background geometry. In the following chapter we will analyse the former —we will consider time symmetric initial data sets with completely general intrinsic curvature. The second direction has not been analysed in this thesis, since the introduction of a non-umbilical extrinsic curvature leads to much more coupling in the linearised equations, and indeed the analysis is likely to be highly dependent on the choice of auxiliary system of equations.

Chapter 5

The Friedrich–Butscher method applied to more general backgrounds

In the previous chapter, the Friedrich–Butscher method was applied to a particular family of background initial data. The aim of the present chapter is to explore the limitations of the method, and to make inroads towards identifying a broader class of admissible background initial data. Again, we shall restrict to closed $\mathcal{S}$. In doing so, we aim to better isolate the geometric content of the problem and identify the key structural features of the equations, along with the potential obstructions.

In order to simplify the analysis, we will also restrict to time symmetric background initial data sets —i.e. constant scalar curvature Riemannian manifolds. Accordingly, the goal of this chapter is to identify the obstructions to using the Friedrich–Butscher method and to give sufficient conditions on the intrinsic background geometry under which the obstructions trivialise and the method can therefore be implemented. As in the previous chapter, these obstructions include the existence of conformal Killing vectors and tracefree Codazzi tensors. The conditions we identify (see (C1)–(C4), below) are assumptions on the spectral properties of a number of elliptic operators defined over $\mathcal{S}$ —see Section 5.1.2 for the definitions of $\mathring{\mathcal{P}}^{(0)}$ and $\mathring{\mathcal{P}}^{(1)}$.

The main result of this chapter (see Theorem 3) is stated informally as follows:

Theorem. Let $(\mathcal{S}, \mathring{\boldsymbol{h}})$ be a time symmetric initial data set satisfying the following conditions

$$
\begin{aligned}
&(C1): \quad (\mathring{\Delta} + \lambda) \quad \text{injective on} \quad C^{\infty}(\mathcal{S}),\\
&(C2): \quad \mathring{\Delta}_Y \quad \text{injective on} \quad \Gamma(\Lambda^1(\mathcal{S})),\\
&(C3): \quad \mathring{P}_L \quad \text{injective on} \quad \Gamma(\mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}}))),\\
&(C4): \quad \mathring{\mathcal{P}}^{(0)}, \mathring{\mathcal{P}}^{(1)} \quad \text{injective on} \quad \Gamma(\mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})) \oplus \Gamma(\Lambda^1(\mathcal{S})).
\end{aligned}
$$

Then, $(\mathcal{S}, \mathring{\boldsymbol{h}})$ admits a family of non-linear perturbative solutions of the extended constraints as in Theorem 2.

It will be shown that these conditions are satisfied by the time symmetric sub-family of (FLRW) background initial data sets considered in the previous chapter. Although it is not done here, it should be straightforward to extend the results of the present chapter to the umbilical case.

In Section 5.3, the restrictions imposed by (C1)–(C4) will be explored further. Some of the conditions will be shown to trivialise for metrics whose curvature is sufficiently *negatively* or *positively-pinched* —see Corollaries 2 and 3. Many of the computations of this chapter were done in xAct; in particular, the linearisations in Section 5.1.1 were performed using the xPert package.

5.1 The ECEs on more general backgrounds

In this section we recall the ECEs, emphasising some of the structural properties that will be important in the remainder of this chapter, and we describe the Friedrich–Butscher in slightly more generality than in the previous Chapter.

5.1.1 Linearisation of the extended constraint equations

Recall that the Friedrich–Butscher method is perturbative, relying on an analysis of a system of linearised auxiliary equations, obtained from the ECEs. It is convenient to describe here the linearisation of the ECEs. Accordingly, consider a one-parameter-family of perturbations (parametrised by a real parameter τ):

$$K(\tau)_{ij} = \mathring{K}_{ij} + \tau\breve{K}_{ij}, \quad \bar{S}(\tau)_{ij} = \mathring{\bar{S}}_{ij} + \tau\breve{\bar{S}}_{ij}, \quad S(\tau)_{ij} = \mathring{S}_{ij} + \tau\breve{S}_{ij}, \quad h(\tau)_{ij} = \mathring{h}_{ij} + \tau\breve{h}_{ij},$$

of a given background solution of the ECEs, $(\mathring{\boldsymbol{K}}, \mathring{\bar{\boldsymbol{S}}}, \mathring{\boldsymbol{S}}, \mathring{\boldsymbol{h}})$. Recall that the linearisation of the Christoffel symbols is given by

$$C(\boldsymbol{\gamma})^i_{jk} \equiv \tfrac{1}{2}(\mathring{D}_j\gamma_k{}^i + \mathring{D}_k\gamma_j{}^i - \mathring{D}^i\gamma_{jk}).$$

As in the previous chapter, whenever we lower/raise indices in linearised expressions, we do so using the background metric and its inverse.

The linearised extended constraint map evaluated at the background solution, denoted by

$$D\Psi[\mathring{\boldsymbol{K}}, \mathring{\bar{\boldsymbol{S}}}, \mathring{\boldsymbol{S}}, \mathring{\boldsymbol{h}}],$$

and acting on the *perturbed quantities* $(\breve{\boldsymbol{K}}, \breve{\bar{\boldsymbol{S}}}, \breve{\boldsymbol{S}}, \breve{\boldsymbol{h}})$, is given by the following Fréchet derivative

$$D\Psi[\mathring{\boldsymbol{K}}, \mathring{\bar{\boldsymbol{S}}}, \mathring{\boldsymbol{S}}, \mathring{\boldsymbol{h}}] \cdot (\breve{\boldsymbol{K}}, \breve{\bar{\boldsymbol{S}}}, \breve{\boldsymbol{S}}, \breve{\boldsymbol{h}}) \equiv \frac{d}{d\tau}\Psi(\boldsymbol{K}(\tau), \bar{\boldsymbol{S}}(\tau), \boldsymbol{S}(\tau), \boldsymbol{h}(\tau))\bigg|_{\tau=0} = \begin{pmatrix} DJ \cdot (\breve{\boldsymbol{K}}, \breve{\bar{\boldsymbol{S}}}, \breve{\boldsymbol{S}}, \breve{\boldsymbol{h}})_{ijk} \\ D\bar{\Lambda} \cdot (\breve{\boldsymbol{K}}, \breve{\bar{\boldsymbol{S}}}, \breve{\boldsymbol{S}}, \breve{\boldsymbol{h}})_{i} \\ D\Lambda \cdot (\breve{\boldsymbol{K}}, \breve{\bar{\boldsymbol{S}}}, \breve{\boldsymbol{S}}, \breve{\boldsymbol{h}})_{i} \\ DV \cdot (\breve{\boldsymbol{K}}, \breve{\bar{\boldsymbol{S}}}, \breve{\boldsymbol{S}}, \breve{\boldsymbol{h}})_{ij} \end{pmatrix} \tag{5.1.1}$$

where the operators $DJ,\ D\bar{\Lambda},\ D\Lambda,\ DV$ act as follows

$$\begin{aligned} DJ \cdot (\breve{\boldsymbol{K}}, \breve{\bar{\boldsymbol{S}}}, \breve{\boldsymbol{S}}, \breve{\boldsymbol{h}})_{ijk} \equiv{}& \mathring{D}_i\breve{K}_{jk} - \mathring{D}_j\breve{K}_{ik} - \mathring{K}_j{}^l C(\boldsymbol{\gamma})_{lik} + \mathring{K}_i{}^l C(\boldsymbol{\gamma})_{ljk} \\ & - \mathring{\epsilon}_{ijl}\breve{\bar{S}}_k{}^l + \mathring{\epsilon}_{ijm}\mathring{\bar{S}}_k{}^l\gamma_l{}^m - \tfrac{1}{2}\mathring{\epsilon}_{ijl}\mathring{\bar{S}}_k{}^l\gamma^m{}_m, \end{aligned} \tag{5.1.2}$$

$$
\begin{aligned}
D\bar{\Lambda}\cdot(\breve{\boldsymbol{K}},\breve{\bar{\boldsymbol{S}}},\breve{\boldsymbol{S}},\breve{\boldsymbol{h}})_l \equiv{}& \mathring{D}_i\breve{\bar{S}}_l{}^i - \gamma^{ij}\mathring{D}_j\mathring{\bar{S}}_{li} - \mathring{\bar{S}}_l{}^i\mathring{D}_j\gamma_i{}^j + \tfrac{1}{2}\mathring{\bar{S}}_l{}^i\mathring{D}_i\gamma^j{}_j \\
&+ \mathring{\epsilon}_{ljk}\mathring{K}^{ij}\breve{S}_i{}^k - \mathring{S}^{ij}\mathring{\epsilon}_{ljk}\breve{K}_i{}^k + \mathring{S}^{ij}\mathring{\epsilon}_{ljm}\mathring{K}^{km}\gamma_{ik} - \mathring{S}^{ij}\mathring{\epsilon}_{lkm}\mathring{K}_i{}^k\gamma_j{}^m \\
&+ \mathring{S}^{ij}\mathring{\epsilon}_{ljm}\mathring{K}_i{}^k\gamma_k{}^m - \tfrac{1}{2}\mathring{S}^{ij}\mathring{\epsilon}_{ljk}\mathring{K}_i{}^k\gamma^m{}_m - \tfrac{1}{2}\mathring{\bar{S}}^{ij}\mathring{D}_l\gamma_{ij}, \qquad (5.1.3)
\end{aligned}
$$

$$
\begin{aligned}
D\Lambda\cdot(\breve{\boldsymbol{K}},\breve{\bar{\boldsymbol{S}}},\breve{\boldsymbol{S}},\breve{\boldsymbol{h}})_i \equiv{}& \mathring{D}_j\breve{S}_i{}^j - \gamma^{jk}\mathring{D}_k\mathring{S}_{ij} - \mathring{S}_i{}^jC(\boldsymbol{\gamma})_j{}^k{}_k \\
&+ \mathring{\epsilon}_{ikl}\bar{S}^{jk}\breve{K}_j{}^l - \mathring{\epsilon}_{ikl}\mathring{K}^{jk}\breve{\bar{S}}_j{}^l + \mathring{\epsilon}_{ikm}\mathring{K}^{jk}\bar{S}^{lm}\gamma_{jl} - \mathring{\epsilon}_{ilm}\mathring{K}^{jk}\bar{S}_j{}^l\gamma_k{}^m \\
&+ \mathring{\epsilon}_{ikm}\mathring{K}^{jk}\bar{S}_j{}^l\gamma_l{}^m - \tfrac{1}{2}\mathring{\epsilon}_{ikl}\mathring{K}^{jk}\bar{S}_j{}^l\gamma^m{}_m - \tfrac{1}{2}\mathring{S}^{jk}\mathring{D}_i\gamma_{jk}, \qquad (5.1.4)
\end{aligned}
$$

$$
\begin{aligned}
DV\cdot(\breve{\boldsymbol{K}},\breve{\bar{\boldsymbol{S}}},\breve{\boldsymbol{S}},\breve{\boldsymbol{h}})_{ij} \equiv{}& -\tfrac{1}{2}\mathring{\Delta}\gamma_{ij} + \tfrac{1}{2}\mathring{D}_k\mathring{D}_i\gamma_j{}^k + \tfrac{1}{2}\mathring{D}_k\mathring{D}_j\gamma_i{}^k - \tfrac{1}{2}\mathring{D}_i\mathring{D}_j\gamma^k{}_k \\
&- \tfrac{2}{3}\lambda\gamma_{ij} - \breve{S}_{ij} + \mathring{K}^k{}_k\breve{K}_{ij} - \mathring{K}_j{}^k\breve{K}_{ik} - \mathring{K}_i{}^k\breve{K}_{jk} \\
&+ \mathring{K}_{ij}\breve{K}^k{}_k + \mathring{K}_i{}^k\mathring{K}_j{}^l\gamma_{kl} - \mathring{K}_{ij}\mathring{K}^{kl}\gamma_{kl}. \qquad (5.1.5)
\end{aligned}
$$

The above were computed using the xPert package in xAct.

Remark 25. Note that for simplicity of notation, we have omitted reference to the background solution of the ECEs in writing the above operators as $DJ,\ D\bar{\Lambda},\ D\Lambda,\ DV$.

For the purposes of this chapter, we will restrict to time symmetric background initial data, $\mathring{K}_{ij} = 0$ (implying $\mathring{\bar{S}}_{ij} = 0$ by the Codazzi–Mainardi equation), for which the linearised operators reduce to

$$DJ\cdot(\breve{\boldsymbol{K}},\breve{\bar{\boldsymbol{S}}},\breve{\boldsymbol{S}},\breve{\boldsymbol{h}})_{ijk} = \mathring{D}_i\breve{K}_{jk} - \mathring{D}_j\breve{K}_{ik} - \mathring{\epsilon}_{ijl}\breve{\bar{S}}_k{}^l, \qquad (5.1.6a)$$
$$D\bar{\Lambda}\cdot(\breve{\boldsymbol{K}},\breve{\bar{\boldsymbol{S}}},\breve{\boldsymbol{S}},\breve{\boldsymbol{h}})_l = \mathring{D}_i\breve{\bar{S}}_l{}^i + \mathring{\epsilon}_{ljk}\breve{K}^{ij}\mathring{S}_i{}^k, \qquad (5.1.6b)$$
$$D\Lambda\cdot(\breve{\boldsymbol{K}},\breve{\bar{\boldsymbol{S}}},\breve{\boldsymbol{S}},\breve{\boldsymbol{h}})_i = \mathring{D}_j\breve{S}_i{}^j - \mathring{S}_i{}^jC(\boldsymbol{\gamma})_j{}^k{}_k - \tfrac{1}{2}\mathring{S}^{jk}\mathring{D}_i\gamma_{jk} - \gamma^{jk}\mathring{D}_k\mathring{S}_{ij}, \qquad (5.1.6c)$$
$$DV\cdot(\breve{\boldsymbol{K}},\breve{\bar{\boldsymbol{S}}},\breve{\boldsymbol{S}},\breve{\boldsymbol{h}})_{ij} = -\tfrac{1}{2}\mathring{\Delta}\gamma_{ij} + \tfrac{1}{2}\mathring{D}_k\mathring{D}_i\gamma_j{}^k + \tfrac{1}{2}\mathring{D}_k\mathring{D}_j\gamma_i{}^k - \tfrac{1}{2}\mathring{D}_i\mathring{D}_j\gamma^k{}_k - \tfrac{2}{3}\lambda\gamma_{ij} - \breve{S}_{ij}. \qquad (5.1.6d)$$

Note that, since the background initial data set is time symmetric, we have

$$\mathring{r}_{ij} = \tfrac{2}{3}\lambda\mathring{h}_{ij} + \mathring{S}_{ij},$$

as a consequence of the Gauss–Codazzi equation.

5.1.2 A family of first-order elliptic operators

In this section, we introduce a family of operators which will be important for the following analysis. For a given Riemannian metric, $\boldsymbol{h}$, define the family of linear operators

$$\mathcal{P}^{(\alpha)} : \mathscr{S}_0^2(\mathcal{S};\boldsymbol{h}) \oplus \Lambda^1(\mathcal{S}) \longrightarrow \mathscr{S}_0^2(\mathcal{S};\boldsymbol{h}) \oplus \Lambda^1(\mathcal{S}),$$

for constant $\alpha \in \mathbb{R}$, as follows

$$\mathcal{P}^{(\alpha)} : \begin{pmatrix} Y_{ij} \\ X_i \end{pmatrix} \longmapsto \begin{pmatrix} 2\mathcal{R}(\boldsymbol{Y})_{ij} + L(\boldsymbol{X})_{ij} \\ -2\delta(\boldsymbol{Y})_i - 2\alpha\,\mathrm{curl}(\boldsymbol{X})_i \end{pmatrix}.$$

We shall see that the operators $\mathcal{P}^{(0)}$, $\mathcal{P}^{(1)}$ appear in the linearisations of the auxiliary equations, both in the direction of the determined fields (see Lemma 11) and in the direction of the free data (see the proof of Theorem 3). The question of their invertibility therefore underpins both the problem of constructing candidate solutions and the problem of the *degeneracy* of the free data —i.e. whether or not two distinct choices of free data give rise to the same solution of the ECEs. In particular, we see that if $\boldsymbol{h}$ admits a tracefree Codazzi tensor, Y_{ij} say, then $(\boldsymbol{Y},\boldsymbol{0})$ lies in the kernel of $\mathcal{P}^{(\alpha)}$ for any choice of $\alpha\in\mathbb{R}$, since

$$\mathcal{P}^{(\alpha)}\begin{pmatrix} Y_{ij} \\ \boldsymbol{0}\end{pmatrix} = \begin{pmatrix} 2\mathcal{R}(\boldsymbol{Y})_{ij} \\ -2\delta(\boldsymbol{Y})_i\end{pmatrix} = \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{pmatrix}$$

—see Remark 3. On the other hand, restricting attention to $\mathcal{P}^{(0)}$, we see that if $\boldsymbol{h}$ admits a conformal Killing vector, X^i say, then $(\boldsymbol{0},\boldsymbol{X})$ lies in the kernel of $\mathcal{P}^{(0)}$, since

$$\mathcal{P}^{(0)}\begin{pmatrix} \boldsymbol{0} \\ X_i\end{pmatrix} = \begin{pmatrix} L(\boldsymbol{X})_{ij} \\ \boldsymbol{0}\end{pmatrix} = \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{pmatrix}.$$

Therefore, we see that injectivity of $\mathring{\mathcal{P}}^{(0)}$ requires (at least) that $\mathring{\boldsymbol{h}}$ admit neither non-trivial tracefree Codazzi tensor fields nor non-trivial conformal Killing vector fields. The non-existence of such tensor fields was already seen to be a necessary condition for invertibility of the linearised auxiliary ECE map (at least when restricting to umbilical background initial data sets) in the previous chapter —see Section 4.1.4. In Chapter 6 the connection between $\mathring{\mathcal{P}}^{(0)}$ and the linearised Codazzi–Mainardi equation will be explained in further detail —see Section 6.1.2.

Lemma 9. The linear operator $\mathcal{P}^{(\alpha)}$, $\alpha\in\mathbb{R}$, is a formally self-adjoint first-order differential operator, which is moreover elliptic for $\alpha\neq-1$.

Proof. The domain and codomain of $\mathcal{P}^{(\alpha)}$ at each $p\in\mathcal{S}$ are isomorphic as vector spaces so it suffices to show that $\mathcal{P}^{(\alpha)}$ is overdetermined elliptic —that is to say, that the symbol map

$$\sigma_\eta(\mathcal{P}^{(\alpha)}):\begin{pmatrix} Y_{ij}\\ X_i\end{pmatrix}\longmapsto\begin{pmatrix} 2\epsilon_{kl(i}\eta^k Y^l{}_{j)}+\eta_i X_j+\eta_j X_i-\frac{2}{3}\eta^k X_k h_{ij}\\ \eta^k Y_{ki}+\alpha\epsilon_{ijk}\eta^j X^k\end{pmatrix}$$

is injective for each $\eta_i\neq0$. Contracting the first component of $\sigma_\eta[\mathcal{P}^{(\alpha)}](\boldsymbol{Y},\boldsymbol{X})=0$ with η^i:

$$\begin{aligned}
0&=\epsilon_{kli}\eta^i\eta^k Y^l{}_j+\epsilon_{klj}\eta^k\eta^i Y^l{}_i+|\boldsymbol{\eta}|^2X_j+\eta_j\eta^iX_i-\tfrac{2}{3}\eta_j\eta^kX_k\\
&=-\alpha\epsilon_{klj}\epsilon^l{}_{mn}\eta^k\eta^mX^n+|\boldsymbol{\eta}|^2X_j+\tfrac{1}{3}\eta_j\eta^iX_i\\
&=-\alpha\eta_j\eta^kX_k+\alpha|\boldsymbol{\eta}|^2X_j+|\boldsymbol{\eta}|^2X_j+\tfrac{1}{3}\eta_j\eta^iX_i\\
&=(-\alpha+\tfrac{1}{3})\eta_j\eta^kX_k+(\alpha+1)|\boldsymbol{\eta}|^2X_j
\end{aligned}$$

Contracting with η^j, we find that $|\boldsymbol{\eta}|^2\eta^kX_k=0$ and therefore we have $\eta^kX_k=0$. Substituting into the above, we find that for $\alpha\neq-1$, $|\boldsymbol{\eta}|^2X_j=0$ and hence $X_j=0$ for $\eta_i\neq0$. Substituting back into the first component of $\sigma_\eta[\mathcal{P}^{(\alpha)})](\boldsymbol{Y},\boldsymbol{X})=0$, we have

$$\epsilon_{kl(i}\eta^kY^l{}_{j)}=0,\quad \text{and}\quad \eta^kY_{ki}=0,$$

or, equivalently,

$$\eta_iY_{jk}-\eta_jY_{ik}=0$$

—i.e. the vanishing of the symbol of $\sigma_\eta[\mathcal{D}]$. Contracting with η^i and using $\eta^iY_{ik}=0$, $|\boldsymbol{\eta}|^2Y_{jk}=0$

and hence $Y_{jk} = 0$. That is to say, $\mathcal{D}|_{\mathscr{S}(\mathcal{S},\boldsymbol{h})}$ is overdetermined elliptic, as shown in Lemma 4 of Section 4.1.1. Therefore, for $\alpha \neq -1$, the operator $\mathcal{P}^{(\alpha)}$ is overdetermined elliptic and hence determined (first-order) elliptic. Formal adjointness follows from the fact that $\mathcal{R}^* = \mathcal{R}$, $\mathrm{curl}^* = \mathrm{curl}$ and $L^* = -2\delta$. □

Remark 26. For completeness, note that $\mathcal{P}^{(-1)}$ is *not* elliptic —given $\eta_i \neq 0$, take any $X_k \neq 0$ for which $X^k\eta_k = 0$ and consider

$$Y_{ij} = \frac{2}{|\boldsymbol{\eta}|^2}\eta_{(i}\epsilon_{j)kl}\eta^k X^l.$$

We find that (X_i, Y_{ij}) is in the kernel of the symbol map $\sigma_\eta[\mathcal{P}^{(-1)}]$.

It will be convenient also to define the operators $\mathcal{K}^{(\alpha)} : \mathcal{J}(\mathcal{S}) \to \mathscr{S}^2_0(\mathcal{S},\boldsymbol{h}) \oplus \Lambda^1(\mathcal{S})$ as follows

$$\mathcal{K}^{(\alpha)}(\boldsymbol{J}) \equiv \begin{pmatrix} 2\mathcal{D}^*(\boldsymbol{J})_{ij} \\ -2\epsilon_{ljk}D^j J^{kl}{}_i + 2(1-\alpha)\epsilon_{ijk}D^j J^{kl}{}_l \end{pmatrix}. \tag{5.1.7}$$

Remark 27. Note that in the previous chapter we defined the operator

$$\mathcal{K}_{\boldsymbol{h}}(\boldsymbol{J}) = \begin{pmatrix} \mathring{\mathcal{D}}^*(\boldsymbol{J})_{ij} \\ \epsilon^{ijk}D_i J_{jkl} \end{pmatrix},$$

for use in the sufficiency argument. It is clear that $\mathring{\mathcal{K}}(\boldsymbol{J}) = 0$ is equivalent to $\mathring{\mathcal{K}}^{(0)}(\boldsymbol{J}) = 0$.

The following lemma shows that $\mathcal{P}^{(\alpha)}$ and $\mathcal{K}^{(\alpha)}$ are really equivalent under the described isomorphism of vector spaces:

Lemma 10. The operators $\mathcal{P}^{(\alpha)}$, $\mathcal{K}^{(\alpha)}$ are related by the Jacobi decomposition

$$\boldsymbol{J}(\boldsymbol{F},\boldsymbol{A})_{ijk} \equiv \tfrac{1}{2}(\epsilon_{ij}{}^l F_{lk} + A_i h_{jk} - A_j h_{ik})$$

as follows

$$\mathcal{P}^{(\alpha)}(\boldsymbol{F},\boldsymbol{A}) \equiv \mathcal{K}^{(\alpha)}(\boldsymbol{J}(\boldsymbol{F},\boldsymbol{A})).$$

Equivalently,

$$\mathcal{K}^{(\alpha)}(\boldsymbol{J}) \equiv \mathcal{P}^{(\alpha)}(\boldsymbol{F}(\boldsymbol{J}),\boldsymbol{A}(\boldsymbol{J}))$$

where $\boldsymbol{J} \mapsto (\boldsymbol{F}(\boldsymbol{J}),\boldsymbol{A}(\boldsymbol{J}))$ is the inverse of the map $(\boldsymbol{F},\boldsymbol{A}) \mapsto \boldsymbol{J}(\boldsymbol{F},\boldsymbol{A})$, given by

$$F(\boldsymbol{J})_{ij} = \epsilon_{kl(i}J^{kl}{}_{j)}, \qquad A(\boldsymbol{J})_i = h^{jk}J_{ijk}.$$

Remark 28. It is clear that $\mathcal{K}^{(\alpha)}$ inherits ellipticity from $\mathcal{P}^{(\alpha)}$ for $\alpha \neq -1$. Moreover, clearly $\mathcal{P}^{(\alpha)}$ is injective if and only if $\mathcal{K}^{(\alpha)}$ is injective. We will pass freely between $\mathcal{P}^{(\alpha)}$ and $\mathcal{K}^{(\alpha)}$ when it is convenient to do so.

Remark 29. It is important to note, for later use, that in order for the operator $\mathcal{P}^{(0)}$ to be injective, $\boldsymbol{h}$ must not admit conformal Killing vectors or tracefree Codazzi tensors —recall that $\mathcal{D}(\boldsymbol{Y})_{ijk} = 0$ if and only if $\mathcal{R}(\boldsymbol{Y})_{ij} = 0$ and $\delta(\boldsymbol{Y})_i = 0$. It is not clear for which $\boldsymbol{h}$, if any, this is a *sufficient* condition for injectivity of $\mathcal{P}^{(0)}$. We will return to the issue of the injectivity of $\mathcal{P}^{(\alpha)}$, more generally, in Section 5.3.3.

5.1.3 Adapting the Friedrich–Butscher method to more general backgrounds

Recall that, roughly speaking, the Friedrich–Butscher method consists of solving an elliptic *auxiliary* system of equations ($\widetilde{\Psi} = 0$, below) which is obtained from the ECEs via an elliptic reduction and the identification of suitable determined and freely-prescribed fields through the use of appropriate ansatz.

Decomposition of time symmetric backgrounds

Recall that the method presented in the previous chapter made use of the following ansatz

$$K_{ij} = \chi_{ij} + \tfrac{1}{3}\phi\mathring{h}_{ij}, \tag{5.1.8a}$$

$$S_{ij} = \Pi_{\boldsymbol{h}}(\mathring{L}(\boldsymbol{X})_{ij} + T_{ij}), \tag{5.1.8b}$$

$$\bar{S}_{ij} = \Pi_{\boldsymbol{h}}(\mathring{L}(\bar{\boldsymbol{X}})_{ij} + \bar{T}_{ij}), \tag{5.1.8c}$$

with $\chi_{ij} \in \mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})$ and $\bar{T}_{ij},\ T_{ij} \in \mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}})$. The quantities $(\chi_{ij}, \bar{X}_i, X_i, h_{ij})$ are the determined fields and $(\phi, \bar{T}_{ij}, T_{ij})$ are the free data.

Any time symmetric background solution may be written in the form of ansatz (5.1.8a)–(5.1.8c), with

$$\mathring{\chi}_{ij} = 0, \qquad \mathring{\bar{X}}_i = 0, \qquad \mathring{X}_i = 0$$

and free data

$$\mathring{\phi} = 0, \quad \mathring{\bar{T}}_{ij} = 0, \quad \mathring{T}_{ij} = \mathring{S}_{ij} = \mathring{r}_{\{ij\}}.$$

To see this, first note that the projection operator is of course simply the identity operator for $h = \mathring{h}_{ij}$. Now, $\mathring{K}_{ij} = 0$ implies $\mathring{\bar{S}}_{ij} = 0$ by the Codazzi–Mainardi equation (see (3.3.13b)) so it follows from the Splitting Lemma (Lemma 3) that $\mathring{\bar{T}}_{ij} = 0$ and $\mathring{L}(\mathring{\bar{\boldsymbol{X}}})_{ij} = 0$. Thus, it is consistent to take $\mathring{\bar{X}}_i = 0$ —later we will restrict to a class of background metric which do not admit conformal Killing fields, in which case $\mathring{\bar{X}}_i = 0$ will follow automatically. Moreover, the Gauss–Codazzi equation implies (given $\mathring{K}_{ij} = 0$) that $\mathring{S}_{ij}$ is nothing other than the tracefree Ricci curvature —see (3.3.13a). Then, since we have $\mathring{r} = 2\lambda$, it follows from the contracted Bianchi identity that $\mathring{\delta}(\mathring{\boldsymbol{S}})_i = 0$ and hence the vector part of the York split vanishes ($\mathring{\bar{X}}_i = 0$), and so that $\mathring{T}_{ij} = \mathring{S}_{ij}$.

As in the previous chapter, having distinguished the freely-prescribed and determined fields, there two *directions* in which one can linearise. For time symmetric backgrounds, linearising in the direction of the determined fields corresponds to setting

$$\breve{K}_{ij} = \sigma_{ij}, \quad \breve{\bar{S}}_{ij} = \mathring{L}(\bar{\boldsymbol{\xi}})_{ij}, \quad \breve{S}_{ij} = \mathring{L}(\boldsymbol{\xi})_{ij} + \tfrac{1}{3}(\mathring{S}^{kl}\gamma_{kl})\mathring{h}_{ij}, \quad \breve{h}_{ij} = \gamma_{ij},$$

where σ_{ij} is again taken to be $\mathring{\boldsymbol{h}}-$tracefree. Note the difference in the perturbations of the electric and magnetic parts, which are obtained by linearising the ansatz (4.1.3b) and (4.1.3c). This is a result of the fact that $\mathring{\bar{S}}_{ij} = 0$ which is a consequence of the assumed time symmetry of the background, while $\mathring{S}_{ij}$ (coinciding with the tracefree Ricci curvature) is kept general.

It will be convenient to define the following tensor fields

$$\begin{pmatrix} \breve{J}_{ijk} \\ \breve{\bar{\Lambda}}_i \\ \breve{\Lambda}_i \\ \breve{V}_{ij} \end{pmatrix} \equiv \begin{pmatrix} DJ \cdot (\boldsymbol{\sigma}, \mathring{L}(\bar{\boldsymbol{\xi}}), \breve{\boldsymbol{S}}, \boldsymbol{\gamma}) \\ D\bar{\Lambda} \cdot (\boldsymbol{\sigma}, \mathring{L}(\bar{\boldsymbol{\xi}}), \breve{\boldsymbol{S}}, \boldsymbol{\gamma}) \\ D\Lambda \cdot (\boldsymbol{\sigma}, \mathring{L}(\bar{\boldsymbol{\xi}}), \breve{\boldsymbol{S}}, \boldsymbol{\gamma}) \\ DV \cdot (\boldsymbol{\sigma}, \mathring{L}(\bar{\boldsymbol{\xi}}), \breve{\boldsymbol{S}}, \boldsymbol{\gamma}) \end{pmatrix},$$

representing the linearisation of the extended constraint map in the direction of the determined fields $(\boldsymbol{\sigma}, \bar{\boldsymbol{X}}, \boldsymbol{X}, \boldsymbol{\gamma})$. The explicit expressions are given by evaluating (5.1.6a)–(5.1.6d).

The auxiliary ECE map and its linearisation on more general backgrounds

The auxiliary system used in this chapter will differ slightly from that of the previous chapter in the construction of the De Turck vector (see Remark 6). Define the *generalised De Turck covector*:

$$Q^{\boldsymbol{X}}(\boldsymbol{h})_i \equiv Q(\boldsymbol{h})_i + 2(\mathring{X}_i - X_i),$$

where $Q(\boldsymbol{h})_i$ is the standard De Turck vector (see the previous chapter) and X_i is the covector appearing in the electric tensor ansatz (5.1.8b). The reason for generalising the De Turck vector as such will become apparent when we come to linearise the resulting gauge-reduced Gauss–Codazzi equation. We will sometimes suppress the dependence on the metric and simply write $Q_i^{\boldsymbol{X}}$. Note that $Q^{\mathring{\boldsymbol{X}}}(\mathring{\boldsymbol{h}})_i = 0$. Since we are restricting to time symmetric data, it follows (see the previous section) that $\mathring{X}_i = 0$. Linearising $Q^{\boldsymbol{X}}(\boldsymbol{h})$ in the direction of the determined fields, one obtains

$$\begin{aligned} \breve{Q}(\boldsymbol{\gamma}, \boldsymbol{\xi})_i &\equiv \frac{d}{d\tau} Q^{\mathring{\boldsymbol{X}}+\tau\boldsymbol{\xi}}(\mathring{\boldsymbol{h}} + \tau\boldsymbol{\gamma})_i \bigg|_{\tau=0} \\ &= \frac{d}{d\tau} Q(\mathring{\boldsymbol{h}} + \tau\boldsymbol{\gamma})_i \bigg|_{\tau=0} - 2\frac{d}{d\tau}(\mathring{\boldsymbol{X}} + \tau\boldsymbol{\xi} - \mathring{\boldsymbol{X}})_i \bigg|_{\tau=0} \\ &= \mathring{\delta}(\boldsymbol{\gamma})_i - \tfrac{1}{2}(d\gamma)_i - 2\xi_i \\ &= \mathring{B}(\boldsymbol{\gamma})_i - 2\xi_i. \end{aligned}$$

Here, we are defining the *Bianchi operator* $B : \mathscr{S}^2(\mathcal{S}) \to \Lambda^1(\mathcal{S})$ as follows

$$B(\boldsymbol{\gamma})_i \equiv C(\boldsymbol{\gamma})_{ij}{}^j = \delta(\boldsymbol{\gamma})_i - \tfrac{1}{2} d(\mathrm{tr}_{\boldsymbol{h}}\boldsymbol{\gamma})_i.$$

Let us also define the operator $\delta^*_{\boldsymbol{h}} : \Lambda^1(\mathcal{S}) \to \mathscr{S}^2(\mathcal{S})$, acting as

$$\delta^*_{\boldsymbol{h}}(\boldsymbol{X})_{ij} \equiv -\tfrac{1}{2}(D_i X_j + D_j X_i)$$

—i.e. the formal L^2-adjoint of $\delta_{\boldsymbol{h}} : \mathscr{S}^2(\mathcal{S}) \to \Lambda^1(\mathcal{S})$. It is useful to note that

$$\frac{d}{d\tau} \delta^*_{\boldsymbol{h}}(Q^{\mathring{\boldsymbol{X}}+\tau\boldsymbol{\xi}}(\mathring{\boldsymbol{h}} + \tau\boldsymbol{\gamma}))_{ij} \bigg|_{\tau=0} = \mathring{\delta}^*(\breve{Q}(\boldsymbol{\gamma}, \boldsymbol{\xi}))_{ij} = \mathring{\delta}^* \circ B(\boldsymbol{\gamma})_{ij} - 2\mathring{\delta}^*(\boldsymbol{\xi})_{ij}, \tag{5.1.9}$$

where we are using the fact that $Q^{\mathring{\boldsymbol{X}}}(\mathring{\boldsymbol{h}})_i = 0$. We define the *reduced Gauss–Codazzi zero quantity* as

$$\widetilde{V}_{ij} \equiv V_{ij} + \delta^*_{\boldsymbol{h}}(Q^{\boldsymbol{X}}(\boldsymbol{h}))_{ij}. \tag{5.1.10}$$

The auxiliary extended constraint map is then as follows

$$\widetilde{\Psi}(\boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}, \boldsymbol{h}; \phi, \bar{\boldsymbol{T}}, \boldsymbol{T}) \equiv \begin{pmatrix} \mathring{\mathcal{D}}^*(\boldsymbol{J})_{ij} \\ \bar{\Lambda}_i \\ \Lambda_i \\ \widetilde{V}_{ij} \end{pmatrix},$$

considered as a second-order differential operator, depending on $(\phi, \bar{\boldsymbol{T}}, \boldsymbol{T})$, and acting on the determined fields, $(\boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}, \boldsymbol{h})$ —it is understood here that we should substitute the ansatz (5.1.8a)–(5.1.8c) for $\boldsymbol{K}, \bar{\boldsymbol{S}}, \boldsymbol{S}$ wherever they appear in the zero-quantities on the right-hand-side.

Let us compute the linearisation of the auxiliary equation for the metric. First recall that the linearisation of the Ricci curvature tensor at $\mathring{\boldsymbol{h}}$ is given (now in terms of $\mathring{\delta}^*$) by

$$\begin{aligned} D\mathrm{Ric}(\mathring{\boldsymbol{h}}) \cdot \gamma_{ij} &= -\tfrac{1}{2}\mathring{\Delta}\gamma_{ij} + \tfrac{1}{2}\mathring{D}_k\mathring{D}_i\gamma_j{}^k + \tfrac{1}{2}\mathring{D}_k\mathring{D}_j\gamma_i{}^k - \tfrac{1}{2}\mathring{D}_i\mathring{D}_j\gamma^k{}_k \\ &= \tfrac{1}{2}\mathring{\Delta}_L\gamma_{ij} - \mathring{\delta}^* \circ \mathring{B}(\boldsymbol{\gamma})_{ij}. \end{aligned}$$

Hence,

$$\begin{aligned} \breve{V}_{ij} &= \tfrac{1}{2}\mathring{\Delta}_L\gamma_{ij} - \mathring{\delta}^* \circ B(\boldsymbol{\gamma})_{ij} - \tfrac{2}{3}\lambda\gamma_{ij} - \mathring{L}(\boldsymbol{\xi})_{ij} - \tfrac{1}{3}(\mathring{S}^{kl}\gamma_{kl})\mathring{h}_{ij} \\ &= \tfrac{1}{2}\mathring{P}_L\gamma_{ij} - \mathring{\delta}^* \circ B(\boldsymbol{\gamma})_{ij} - \mathring{L}(\boldsymbol{\xi})_{ij} - \tfrac{1}{3}(\mathring{S}^{kl}\gamma_{kl})\mathring{h}_{ij}. \end{aligned}$$

Combining this with equation (5.1.9), we obtain

$$\begin{aligned} D_v\widetilde{V} \cdot (\boldsymbol{\xi}, \boldsymbol{\gamma})_{ij} &= \breve{V}_{ij} + \mathring{\delta}(\breve{Q}(\boldsymbol{\gamma}, \boldsymbol{\xi}))_{ij} \\ &= \tfrac{1}{2}\mathring{\Delta}_L\gamma_{ij} - \mathring{\delta}^* \circ \mathring{B}(\boldsymbol{\gamma})_{ij} - \tfrac{2}{3}\lambda\gamma_{ij} - \mathring{L}(\boldsymbol{\xi})_{ij} - \tfrac{1}{3}(\mathring{S}^{kl}\gamma_{kl})\mathring{h}_{ij} + \mathring{\delta}^*(\breve{Q}(\boldsymbol{\gamma}, \boldsymbol{\xi}))_{ij} \\ &= \tfrac{1}{2}\mathring{P}_L\gamma_{ij} - \mathring{L}(\boldsymbol{\xi})_{ij} - \tfrac{1}{3}(\mathring{S}^{kl}\gamma_{kl})\mathring{h}_{ij} + 2\mathring{D}_{(i}\xi_{j)} \\ &= \tfrac{1}{2}\mathring{P}_L\gamma_{ij} + \tfrac{1}{3}(2\mathring{\delta}(\boldsymbol{\xi}) - \mathring{S}^{kl}\gamma_{kl})\mathring{h}_{ij}, \end{aligned}$$

which is manifestly elliptic. Note the cancellation of the term $2\mathring{D}_{(i}\xi_{j)}$, coming from the contribution of the De Turck term, and the $\mathring{L}(\boldsymbol{\xi})-$term, leaving only the $\mathring{\delta}(\boldsymbol{\xi})$ components. It is this last cancellation which motivates the above choice of generalised De Turck covector.

The linearisation of the full map $\widetilde{\Psi}$ in the direction of the determined fields, denoted $D_v\widetilde{\Psi}$, decouples into two maps which we denote $D_v\widetilde{\Psi}_1,\ D_v\widetilde{\Psi}_2$, and which are given by

$$D_v\widetilde{\Psi}_1 \cdot (\boldsymbol{\sigma}, \bar{\boldsymbol{\xi}}) \equiv \begin{pmatrix} \mathring{\mathcal{D}}^*(\dot{\boldsymbol{J}})_{ij} \\ \dot{\bar{\Lambda}}_l \end{pmatrix} \equiv \begin{pmatrix} \mathring{\mathcal{D}}^*(\mathring{\mathcal{D}}(\boldsymbol{\sigma}) - (\mathring{\star}\mathring{L}(\bar{\boldsymbol{\xi}}))_{ij} \\ \mathring{\delta} \circ \mathring{L}(\bar{\boldsymbol{\xi}})_l + \mathring{\epsilon}_{ljk}\sigma^{ij}\mathring{S}_i{}^k \end{pmatrix}, \tag{5.1.11}$$

$$D_v\widetilde{\Psi}_2 \cdot (\boldsymbol{\xi}, \boldsymbol{\gamma}) \equiv \begin{pmatrix} \dot{\Lambda}_i \\ \dot{V}_{ij} - \mathring{D}_{(i}\breve{Q}_{j)} \end{pmatrix} \equiv \begin{pmatrix} \mathring{\delta} \circ \mathring{L}(\boldsymbol{\xi})_i + \tfrac{1}{3}\mathring{D}_i(\mathring{S}^{jk}\gamma_{jk}) - \mathring{S}_i{}^j\mathring{\delta}(\boldsymbol{\gamma})_j + \tfrac{1}{2}\mathring{S}_i{}^j(d\gamma)_j \\ -\tfrac{1}{2}\mathring{S}^{jk}\mathring{D}_i\gamma_{jk} - \gamma^{jk}\mathring{D}_k\mathring{S}_{ij} \\ \tfrac{1}{2}\mathring{P}_L\gamma_{ij} + \tfrac{1}{3}(2\mathring{\delta}(\boldsymbol{\xi}) - \mathring{S}^{kl}\gamma_{kl})\mathring{h}_{ij} \end{pmatrix}. \tag{5.1.12}$$

Although the system $D_v\widetilde{\Psi} = 0$ is much less coupled than in the general (i.e. non-time symmetric) case, it is however more complicated than in the previous chapter (in which the background geometry was Einstein), since we are allowing here for non-trivial tracefree Ricci curvature of the background –i.e. $\mathring{S}_{ij} \not\equiv 0$.

As in the previous chapter, the principal part of $D_v\widetilde{\Psi}$ is again given by

$$\begin{pmatrix} \mathring{\mathcal{D}}^*\circ\mathring{\mathcal{D}} & \mathring{\mathcal{D}}^*(\star\mathring{L}) & 0 & 0 \\ 0 & \mathring{\delta}\circ\mathring{L} & 0 & 0 \\ 0 & 0 & \mathring{\delta}\circ\mathring{L} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\mathring{\Delta} \end{pmatrix}\begin{pmatrix} \sigma_{ij} \\ \bar{\xi}_i \\ \xi_i \\ \gamma_{ij} \end{pmatrix},$$

and $D_v\widetilde{\Psi}$ is again second-order elliptic, and hence Fredholm for compact $\mathcal{S}$. The following lemma will prove useful when we come to analyse the kernel/cokernel of $D_v\widetilde{\Psi}$ in subsequent sections.

Lemma 11. The operator $D_v\widetilde{\Psi}_1$ may be factorised as follows

$$D_u\widetilde{\Psi}_1\cdot(\boldsymbol{\sigma},\boldsymbol{\xi}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\circ\mathring{\mathcal{P}}^{(1)}\circ\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\circ\mathring{\mathcal{P}}^{(0)}\begin{pmatrix} \boldsymbol{\sigma} \\ -2\boldsymbol{\xi} \end{pmatrix}, \tag{5.1.13}$$

from which it follows that the formal adjoint acts on $X_i\in\Lambda^1(\mathcal{S}), Y_{ij}\in\mathscr{S}^2_0(\mathcal{S},\mathring{\boldsymbol{h}})$ as

$$(D_u\widetilde{\Psi}_1)^*\cdot(\boldsymbol{Y},\boldsymbol{X}) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\circ\mathring{\mathcal{P}}^{(0)}\circ\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\circ\mathring{\mathcal{P}}^{(1)}\begin{pmatrix} \boldsymbol{Y} \\ \frac{1}{2}\boldsymbol{X} \end{pmatrix}.$$

Proof. The expression for $D_v\widetilde{\Psi}_1$ follows by direct computation using the identities

$$\begin{aligned} &\mathring{\mathcal{D}}^*\circ\mathring{\mathcal{D}}(\boldsymbol{\sigma})_{ij} = 2\mathring{\mathcal{R}}\circ\mathring{\mathcal{R}}(\boldsymbol{\sigma})_{ij} - \tfrac{1}{2}\mathring{L}\circ\mathring{\delta}(\boldsymbol{\sigma})_{ij}, \\ &\mathring{\mathcal{D}}^*(\star(\mathring{L}(\boldsymbol{\xi}))_{ij} = 2\mathring{\mathcal{R}}\circ\mathring{L}(\boldsymbol{\xi})), \\ &\mathring{\delta}\circ\mathring{\mathcal{R}}(\boldsymbol{\sigma})_i = \tfrac{1}{2}\mathrm{curl}\circ\mathring{\delta}(\boldsymbol{\sigma})_i - \epsilon_{ikl}\mathring{r}_j{}^l\sigma^{jk}. \end{aligned}$$

The computation of the adjoint follows by a straightforward computation using the fact that the operators $\mathcal{P}^{(\alpha)}$ are formally self-adjoint —see Lemma 9. □

Remark 30. The appearance of the operator $\mathring{\mathcal{P}}^{(1)}$ in the above formula for $D_v\widetilde{\Psi}_1$ is due to our choice of auxiliary map $\widetilde{\Psi}$. If, rather than using the auxiliary equation $\mathring{\mathcal{D}}^*(\boldsymbol{J})_{ij}=0$ (as considered in this and the previous Chapter), one were to use a different auxiliary equation, then the formula given in Lemma 11 would of course not hold and the operator $\mathring{\mathcal{P}}^{(1)}$ would not appear. In fact, we show in Chapter 6 that if one is willing to work with mixed-order elliptic systems, then one can avoid the introduction of $\mathring{\mathcal{P}}^{(1)}$ altogether by working directly with the equation $J_{ijk}=0$ which, as shown in Section 6.1.2, can be rendered first-order elliptic.

Since $D_v\widetilde{\Psi}$ decouples into the two maps $D_v\widetilde{\Psi}_1, D_v\widetilde{\Psi}_2$, the formal adjoint of $D_v\widetilde{\Psi}$, denoted $D_v\widetilde{\Psi}^*$, also decouples into the formal adjoints $(D_v\widetilde{\Psi}_1)^*, (D_v\widetilde{\Psi}_2)^*$. The operator $(D_v\widetilde{\Psi}_1)^*$ is given in Lemma 11; it is not advantageous to expand out the given composition of operators. For $(D_v\widetilde{\Psi}_2)^*$, we find

$$(D_v\widetilde{\Psi}_2)^*\begin{pmatrix} \varsigma_i \\ \eta_{ij} \end{pmatrix} = \begin{pmatrix} \mathring{\delta}\circ\mathring{L}(\boldsymbol{\varsigma})_i - \frac{2}{3}(d\eta)_i \\ \frac{1}{2}\mathring{P}_L\eta_{ij} + \frac{1}{2}\mathcal{L}_{\boldsymbol{\varsigma}}\mathring{S}_{ij} - \varsigma^k\mathring{D}_{(i}\mathring{S}_{j)k} - \frac{1}{12}\mathring{S}^{kl}\mathring{L}(\boldsymbol{\varsigma})_{kl}\mathring{h}_{ij} + \frac{1}{6}(\mathring{\delta}(\boldsymbol{\varsigma})-2\eta)\mathring{S}_{ij} \end{pmatrix}, \tag{5.1.14}$$

for arbitrary $\varsigma_i\in\Lambda^1(\mathcal{S})$, $\eta_{ij}\in\mathscr{S}^2(\mathcal{S})$. For later use, we note that we can also linearise the integrability conditions (3.3.12a) and(3.3.12b). In particular, linearising (3.3.12b) in the direction of the determined fields, one obtains

$$\mathring{B}(\breve{\boldsymbol{V}})_j = -\breve{\Lambda}_j + \mathring{K}_j{}^i\breve{J}_i{}^k{}_k + \mathring{K}^{ik}\breve{J}_{jik} - \mathring{K}^i{}_i\breve{J}_j{}^k{}_k, \tag{5.1.15}$$

where we are using the fact that the ECEs are satisfied by the background solution. In the time symmetric case considered here, all terms apart from the $\breve{\Lambda}_j$ term will trivialise. This relation will be used in Section 5.2.2 to prove invertibility of $D_v\widetilde{\Psi}$.

Finally, the linearisation of $\widetilde{\Psi}$ in the direction of the free data (for time-symmetric background data) is given by

$$\begin{aligned} D_u\widetilde{\Psi}(\breve{\phi},\breve{\bar{\boldsymbol{T}}},\breve{\boldsymbol{T}}) &\equiv \frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{\Psi}(\boldsymbol{\chi},\bar{\boldsymbol{X}},\boldsymbol{X},\boldsymbol{h};\ \mathring{\phi}+\tau\breve{\phi},\mathring{\bar{\boldsymbol{T}}}+\tau\breve{\bar{\boldsymbol{T}}},\mathring{\boldsymbol{T}}+\tau\breve{\boldsymbol{T}})\Big|_{\tau=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{\Psi}(\boldsymbol{\chi},\bar{\boldsymbol{X}},\boldsymbol{X},\boldsymbol{h};\ \tau\breve{\phi},\tau\breve{\bar{\boldsymbol{T}}},\mathring{\boldsymbol{S}}+\tau\breve{\boldsymbol{T}})\Big|_{\tau=0} = \begin{pmatrix} \mathring{\mathcal{R}}(\breve{\bar{\boldsymbol{T}}})-\frac{1}{6}\mathring{L}(d\phi)_{jk} \\ \mathring{\delta}(\breve{\bar{\boldsymbol{T}}})_i \\ \mathring{\delta}(\breve{\boldsymbol{T}})_i \\ -\breve{T}_{ij} \end{pmatrix}, \end{aligned} \tag{5.1.16}$$

where we are again using the fact that the ECEs are satisfied for the background solution, so that $J_{ijk},\bar{\Lambda}_i,\Lambda_i,V_{ij}$ vanish at zeroth order in τ. As in the previous chapter, the operator $D_u\widetilde{\Psi}$ will be important when we consider the issue of degeneracy of the free data —see Theorem 3.

5.2 Non-linear perturbations of time symmetric initial data

In this section we identify sufficient conditions for a given time symmetric background initial data set to admit non-linear perturbative solutions of the ECEs via the Friedrich–Butscher method.

5.2.1 The conditions on the background metric

In the following, the operators are as defined in Chapter 4, with respect to the metric $\mathring{\boldsymbol{h}}$:

$$\begin{aligned} &(C1): \quad (\mathring{\Delta}+\lambda) \quad \text{injective on} \quad C^\infty(\mathcal{S}), \\ &(C2): \quad \mathring{\Delta}_Y \quad \text{injective on} \quad \Gamma(\Lambda^1(\mathcal{S})), \\ &(C3): \quad \mathring{P}_L \quad \text{injective on} \quad \Gamma(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}}))), \\ &(C4): \quad \mathring{\mathcal{P}}^{(0)},\mathring{\mathcal{P}}^{(1)} \quad \text{injective on} \quad \Gamma(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}}))\oplus\Gamma(\Lambda^1(\mathcal{S})). \end{aligned}$$

Note that condition (C4) is equivalent to the statement that $\mathcal{K}^{(0)}$ and $\mathcal{K}^{(1)}$ are injective on $\Gamma(\mathcal{J}(\mathcal{S}))$. Condition (C2) is precisely the statement that $\mathring{\boldsymbol{h}}$ admit no *infinitesimal harmonic deformations*. Recall that

$$\mathring{\Delta}_Y(\boldsymbol{V})_i \equiv -\mathring{\delta}(\mathcal{L}_{\boldsymbol{V}}\boldsymbol{h})_i - \tfrac{1}{2}\mathring{D}_i(\mathcal{L}_{\boldsymbol{V}}\boldsymbol{h})_j{}^j.$$

A Killing vector is therefore clearly a special case of an infinitesimal harmonic deformation; in order for condition (C2) to be satisfied, it is a necessary condition that $\mathring{\boldsymbol{h}}$ does not admit any Killing vector fields. In particular, elliptic manifolds[1] necessarily violate (C2) since they have non-trivial groups of isometries —see [67]. Moreover, we note that if $\mathring{\boldsymbol{h}}$ is a (non-Einstein) Ricci soliton metric then it necessarily violates (C2). Recall that a Ricci soliton is a Riemannian manifold $(\mathcal{S},\mathring{\boldsymbol{h}})$ for which there exists a covector field $V_i \not\equiv 0$ and a constant $\tilde{\lambda}$ such that

$$\mathrm{Ric}[\mathring{\boldsymbol{h}}]_{ij} = \tilde{\lambda}\mathring{\boldsymbol{h}} + (\mathcal{L}_{\boldsymbol{V}}\mathring{\boldsymbol{h}})_{ij}, \tag{5.2.1}$$

[1] Recall that a 3-dimensional manifold $(\mathcal{S},\boldsymbol{h})$ is elliptic if it is Einstein with positive curvature.

and describes a *self-similar* solution to the Ricci flow —see [53], for instance. To see that such a metric violates condition (C2), note that (5.2.1) and the contracted Bianchi identity imply

$$-\mathring{\Delta}_Y(\boldsymbol{V})_i = \mathring{\delta}(\mathcal{L}_{\boldsymbol{V}}\mathring{\boldsymbol{h}})_i - \tfrac{1}{2}\mathring{D}_i(\mathcal{L}_{\boldsymbol{V}}\mathring{\boldsymbol{h}})_j{}^j = \mathring{\delta}(\mathring{\boldsymbol{r}})_i - \tfrac{1}{2}(d\mathring{r})_i = 0,$$

so that V_i is necessarily an infinitesimal harmonic deformation for $\mathring{\boldsymbol{h}}$ —see [68] for more details.

The existence of conformal Killing vectors of non-vanishing divergence —i.e. non pure-Killing vectors— is not necessarily ruled out by assuming condition (C2). This may be achieved, however, if one replaces injectivity of $\mathring{\Delta}_Y$ with the stronger assumption that it has strictly positive spectrum i.e. that $\mathring{\Delta}_Y > 0$—we will return to this idea in Section 5.3.

As noted in Remark 29 of Section 5.1.2, (C4) already requires the non-existence of conformal Killing vectors. Condition (C4) will be used three times in the proof of the main result: injectivity of $\mathring{\mathcal{P}}^{(0)}$ will be used in the construction of candidate solutions and in the proof that the map from free data to candidate solutions (given by the IFT) is injective, while injectivity of $\mathring{\mathcal{P}}^{(1)}$ will be used in the sufficiency argument. It is clear from the factorisation of $D_v\widetilde{\Psi}_1$ in Lemma 11 that if $\mathring{\mathcal{P}}^{(0)}$ and $\mathring{\mathcal{P}}^{(1)}$ are injective then so is $D_v\widetilde{\Psi}_1$. The following proposition establishes that the converse is also true, so that condition (C4) is precisely what we need to ensure injectivity of $D_v\widetilde{\Psi}_1$.

Proposition 10. $D_v\widetilde{\Psi}_1$ is injective on $H^2(\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}})) \times H^2(\Lambda^1(\mathcal{S}))$ if and only if condition (C4) holds.

Proof. First note that by elliptic regularity it suffices to restrict to smooth sections.

"⇐": Integrate up $D_v\widetilde{\Psi}_1(\boldsymbol{Y},\boldsymbol{X}) = 0$ using the formula in Lemma 11 and use the assumed injectivity of $\mathring{\mathcal{P}}^{(0)}, \mathring{\mathcal{P}}^{(1)}$ to show that $(\boldsymbol{Y},\boldsymbol{X}) = (\boldsymbol{0},\boldsymbol{0})$.

"⇒": Suppose it is not the case that $\mathring{\mathcal{P}}^{(0)}$ and $\mathring{\mathcal{P}}^{(1)}$ are both injective. We distinguish two sub-cases:

Case 1: $\mathring{\mathcal{P}}^{(0)}$ is not injective: there exists $(\boldsymbol{Y},\boldsymbol{X}) \in \mathscr{S}_0^2(\mathcal{S},\boldsymbol{h}) \oplus \Lambda^1(\mathcal{S})$ with $(\boldsymbol{Y},\boldsymbol{X}) \not\equiv (\boldsymbol{0},\boldsymbol{0})$ such that $\mathring{\mathcal{P}}^{(0)}(\boldsymbol{Y},\boldsymbol{X}) = 0$. Clearly then $D_v\Psi_1(\boldsymbol{Y},\boldsymbol{X}) = 0$ by Lemma 11, implying that $D_v\widetilde{\Psi}_1$ is not injective.

Case 2: $\mathring{\mathcal{P}}^{(0)}$ is injective, but $\mathring{\mathcal{P}}^{(1)}$ is not: there exists $(\boldsymbol{F},\boldsymbol{A}) \in \mathscr{S}_0^2(\mathcal{S},\boldsymbol{h}) \oplus \Lambda^1(\mathcal{S})$ with $(\boldsymbol{F},\boldsymbol{A}) \not\equiv (\boldsymbol{0},\boldsymbol{0})$ such that $\mathring{\mathcal{P}}^{(1)}(\boldsymbol{F},\boldsymbol{A}) = 0$. Consider then $(\boldsymbol{F},2\boldsymbol{A})$; we want to argue that it is in the image of $\mathring{\mathcal{P}}^{(0)}$. Recall from Lemma 9 that $\mathring{\mathcal{P}}^{(0)}$ is elliptic, so by Fredholm theory we have that $\mathring{\mathcal{P}}^{(0)}$ is surjective, since

$$\mathrm{Im}\ \mathring{\mathcal{P}}^{(0)} = (\ker\ (\mathring{\mathcal{P}}^{(0)})^*)^\perp = (\ker\ \mathring{\mathcal{P}}^{(0)})^\perp = \{\boldsymbol{0}\}^\perp,$$

where we are using the self-adjointness of $\mathring{\mathcal{P}}^{(0)}$ and its assumed injectivity. Hence, $(\boldsymbol{F},2\boldsymbol{A}) = \mathring{\mathcal{P}}^{(0)}(\boldsymbol{Y},\boldsymbol{X})$ for some (unique) $(\boldsymbol{Y},\boldsymbol{X}) \in \mathscr{S}_0^2(\mathcal{S},\boldsymbol{h}) \oplus \Lambda^1(\mathcal{S})$ with $(\boldsymbol{Y},\boldsymbol{X}) \not\equiv (\boldsymbol{0},\boldsymbol{0})$. Then, by the formula in Lemma 11, $D_v\widetilde{\Psi}_1(\boldsymbol{Y},\boldsymbol{X}) = 0$ and we conclude again that $D_v\widetilde{\Psi}_1$ is not injective. ☐

Finally, turning to (C3), we remark that the question of whether or not $\mathring{P}_L$ is injective arises naturally in the study of Einstein manifolds; for an Einstein metric $\mathring{\boldsymbol{h}}$, an element $\boldsymbol{\eta} \in \ker \mathring{P}_L \cap \mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}})$ describes an *Einstein deformation* —i.e. a perturbation of the metric which preserves the property of being Einstein "to first order". Note however that here we consider a broader class of background metrics than the Einstein metrics. Also, condition (C3) requires that $\mathring{P}_L$ be injective on the larger space $\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}})$. In general, it is difficult to reduce the problem of injectivity on $\mathscr{S}_0^2(\mathcal{S};\mathring{\boldsymbol{h}})$ to that of injectivity on $\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}})$ unless the background Ricci curvature is covariantly-constant, in which case one can proceed by first constructing an elliptic equation for $\mathring{\delta}(\boldsymbol{\eta})$ —see [69].

5.2.2 Existence of candidate solutions

We fix our functional spaces $\mathcal{X}^s$, $\mathcal{Y}^s$, $\mathcal{Z}^s$, for $s \geq 4$, as in Chapter 4. For smooth background initial data data, it is straightforward to show using the Schauder ring property —see Section 2.3— that, indeed, $D_v\widetilde{\Psi} : \mathcal{X}^s \times \mathcal{Y}^s \to \mathcal{Z}^s$. In this section we show that if conditions (C1)–(C4) are assumed to hold, then the linearised auxiliary map $D_v\widetilde{\Psi}$ is an isomorphism of Banach spaces. As before, an application of the IFT then guarantees the existence of a map ν from the free data to the space of solutions of $\widetilde{\Psi} = 0$.

We will see that, in contrast to the previous chapter, we will need to make use of one of the (linearised) integrability conditions in order to argue injectivity of $D_v\widetilde{\Psi}$. This is necessitated by the additional coupling of the equations. The linearisation of the remaining integrability condition would presumably have a role to play in arguing injectivity of $D_v\widetilde{\Psi}$ in the more general (i.e. non-time symmetric) case. This is not explored here, however.

Lemma 12. Given a time symmetric background initial data set $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathbf{0})$, suppose that γ_{ij}, ξ_i satisfy the linearised equations $\breve{\Lambda}_i \equiv D_v\Lambda_i = 0, D_v\widetilde{V}_{ij} = 0$, then the quantity $\breve{Q}_i = \breve{Q}(\boldsymbol{\gamma}, \boldsymbol{\xi})_i \equiv \mathring{B}(\boldsymbol{\gamma})_i - 2\xi_i$ satisfies the equation

$$\mathring{\Delta}_Y \breve{Q}_i = 0.$$

Hence, if $(\mathcal{S}, \mathring{\boldsymbol{h}})$ satisfies condition (C2), then

$$\mathring{B}(\boldsymbol{\gamma})_i = 2\xi_i.$$

Proof. First note that

$$0 = D_v\widetilde{V}_{ij} = \breve{V}_{ij} + \mathring{\delta}^*(\breve{Q}(\boldsymbol{\gamma}, \boldsymbol{\xi}))_{ij}.$$

Hence,

$$\breve{V}_{ij} = -\mathring{\delta}^*(\breve{Q}(\boldsymbol{\gamma}, \boldsymbol{\xi}))_{ij}.$$

Substituting into the linearised integrability condition (5.1.15), we therefore see that

$$\tfrac{1}{2}\mathring{\Delta}_Y \breve{Q}_j \equiv \mathring{B} \circ \mathring{\delta}^*(\breve{Q})_j = -\breve{\Lambda}_j + \mathring{K}_j{}^i \breve{J}_i{}^k{}_k + \mathring{K}^{ik}\breve{J}_{jik} - \mathring{K}^i{}_i \breve{J}_j{}^k{}_k = 0,$$

where the final equality follows from the fact that (by assumption) the equation $\breve{\Lambda}_i = 0$ is satisfied, and that $\mathring{K}_{ij} = 0$. $\square$

Proposition 11. Suppose $\mathring{\boldsymbol{h}}$ satisfies conditions (C1)–(C4), then the linearised auxiliary ECE map, $D_v\widetilde{\Psi} : \mathcal{Y}^s \to \mathcal{Z}^s$, $s \geq 4$, is an isomorphism of Banach spaces. Hence, the IFT implies that there exists an open neighbourhood $\mathcal{U} \subseteq \mathcal{X}^s$ of $(\mathbf{0}, \mathbf{0}, \mathbf{0})$, an open neighbourhood $\mathcal{W} \subseteq \mathcal{Y}^s$ of $(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathring{\boldsymbol{h}})$ and a smooth map $\nu : \mathcal{U} \to \mathcal{W}$ such that, $\widetilde{\Psi}(\nu(u); u) = 0$ for every $u \in \mathcal{U}$.

Proof. By the Fredholm alternative, we need only show that $D_v\widetilde{\Psi}$ and $(D_v\widetilde{\Psi})^*$ are injective. First note that, by elliptic regularity, it suffices to consider $D_v\widetilde{\Psi}$ and $(D_v\widetilde{\Psi})^*$ acting on smooth fields.

Injectivity: Let us first consider the map $D_v\widetilde{\Psi}_1$. Note that $\mathring{\mathcal{P}}^{(0)}, \mathring{\mathcal{P}}^{(1)}$ are injective, and so from Lemma 11 $D_v\widetilde{\Psi}_1$ is a composition of injective operators and hence is itself injective.

Decomposing the second component of $D_v\widetilde{\Psi}_2 = 0$ into trace and tracefree parts with respect to $\mathring{\boldsymbol{h}}$, we obtain

$$\left(\mathring{\Delta} + \tfrac{4\lambda}{3}\right)\gamma - 4\mathring{\delta}(\boldsymbol{\xi}) + 2\mathring{S}^{ij}\breve{\gamma}_{ij} = 0, \tag{5.2.2a}$$

$$\tfrac{1}{2}\mathring{P}_L \bar{\gamma}_{ij} = 0, \tag{5.2.2b}$$

where $\gamma \equiv \gamma_k{}^k$, $\breve{\gamma}_{ij} \equiv \gamma_{ij} - \frac{1}{3}\gamma\mathring{h}_{ij}$. By condition (C3) and (5.2.2b) we have $\bar{\gamma}_{ij} = 0$. Substituting into (5.2.2a), we obtain

$$\left(\mathring{\Delta} + \tfrac{4\lambda}{3}\right)\gamma - 4\mathring{\delta}(\boldsymbol{\xi}) = 0. \tag{5.2.3}$$

Now, using condition (C2), Lemma 12 implies that

$$\breve{Q}_i \equiv \mathring{\delta}(\boldsymbol{\gamma})_i - \tfrac{1}{2}(d\gamma)_i - 2\xi_i = 0. \tag{5.2.4}$$

Taking the divergence of (5.2.4) and using $\bar{\gamma}_{ij} = 0$, we arrive at

$$\mathring{\Delta}\gamma + 12\mathring{\delta}(\boldsymbol{\xi}) = 0, \tag{5.2.5}$$

which combined with (5.2.3) implies $(\mathring{\Delta} + \lambda)\gamma = 0$. Hence, by condition (C1), we see that $\gamma = 0$. Finally substituting back into (5.2.4), we find that $\xi_i = 0$. Hence, the map $D_v\widetilde{\Psi}_2$ is injective. Since $D_v\widetilde{\Psi}_1$, $D_v\widetilde{\Psi}_2$ are injective, so too is the full linearised auxiliary constraint map $D_v\widetilde{\Psi}$.

Surjectivity: Let us first consider surjectivity of $D_v\widetilde{\Psi}_1$. Using Lemma 11 the map $(D_v\widetilde{\Psi}_1)^*$ is clearly injective since $\mathring{\mathcal{P}}^{(0)}$, $\mathring{\mathcal{P}}^{(1)}$ are, by assumption, injective. Applying the Fredholm alternative (recalling that $D_v\Psi_1$ is elliptic) it follows that $D_v\widetilde{\Psi}_1$ is surjective. Consider now $D_v\widetilde{\Psi}_2$; we want to show injectivity of the adjoint, $(D_v\widetilde{\Psi}_2)^*$. Consider then $(D_v\widetilde{\Psi}_2)^* \cdot (\boldsymbol{\eta}, \boldsymbol{\varsigma}) = 0$, given explicitly by

$$\mathring{\delta} \circ \mathring{L}(\boldsymbol{\varsigma})_i - \tfrac{2}{3}(d\eta)_i = 0, \tag{5.2.6a}$$

$$\tfrac{1}{2}\mathring{P}_L\eta_{ij} + \tfrac{1}{2}\mathcal{L}_{\boldsymbol{\varsigma}}\mathring{S}_{ij} - \varsigma^k\mathring{D}_{(i}\mathring{S}_{j)k} - \tfrac{1}{12}\mathring{S}^{kl}\mathring{L}(\boldsymbol{\varsigma})_{kl}\mathring{h}_{ij} + \tfrac{1}{6}(\mathring{\delta}(\boldsymbol{\varsigma}) - 2\eta)\mathring{S}_{ij} = 0 \tag{5.2.6b}$$

—see (5.1.14). The trace and tracefree parts of (5.2.6b) are given by

$$\left(\mathring{\Delta} + \tfrac{4\lambda}{3}\right)\eta + \tfrac{1}{2}\mathring{S}^{ij}\mathring{L}(\boldsymbol{\varsigma})_{ij} = 0, \tag{5.2.7a}$$

$$\tfrac{1}{2}\mathring{P}_L\bar{\eta}_{ij} + \tfrac{1}{2}\varsigma^k\mathring{D}_k\mathring{S}_{ij} - \varsigma\mathring{D}_{\{i}\mathring{S}_{j\}k} + \mathring{S}_{k\{i}\mathring{D}_{j\}}\varsigma^k + \tfrac{1}{6}(\mathring{\delta}(\varsigma) - 2\eta)\mathring{S}_{ij} = 0, \tag{5.2.7b}$$

where $\eta \equiv \eta_k{}^k$, $\bar{\eta}_{ij} \equiv \eta_{ij} - \frac{1}{3}\eta\mathring{h}_{ij}$. On the other hand, taking the divergence of (5.2.6a) we obtain

$$\begin{aligned}
0 &= \mathring{D}^i\left(\mathring{\delta} \circ \mathring{L}(\boldsymbol{\varsigma})_i - \tfrac{2}{3}(d\eta)_i\right) \\
&= \mathring{D}^i\left(\mathring{\Delta}\varsigma_i + \mathring{S}_i{}^j\varsigma_j + \tfrac{2}{3}\lambda\varsigma_i + \tfrac{1}{3}\mathring{D}_i\mathring{\delta}(\boldsymbol{\varsigma}) - \tfrac{2}{3}(d\eta)_i\right) \\
&= \mathring{D}^i\mathring{\Delta}\varsigma_i + \tfrac{1}{3}\mathring{\Delta}\mathring{\delta}(\boldsymbol{\varsigma}) + \tfrac{1}{2}\mathring{S}^{ij}\mathring{L}(\boldsymbol{\varsigma})_{ij} + \tfrac{2}{3}\lambda\mathring{\delta}(\boldsymbol{\varsigma}) - \tfrac{2}{3}\mathring{\Delta}\eta \\
&= \tfrac{4}{3}\mathring{\Delta}\mathring{\delta}(\boldsymbol{\varsigma}) + \mathring{S}^{ij}\mathring{L}(\boldsymbol{\varsigma})_{ij} + \tfrac{4}{3}\lambda\mathring{\delta}(\boldsymbol{\varsigma}) - \tfrac{2}{3}\mathring{\Delta}\eta,
\end{aligned}$$

where we have repeatedly used the fact that $\mathring{\delta}(\mathring{\boldsymbol{S}})_i = 0$ and the final line follows by commuting the divergence and Laplacian in the preceding line. Substituting for the $\mathring{S}^{ij}\mathring{L}(\boldsymbol{\varsigma})_{ij}$ term using (5.2.7a), we find that

$$(\mathring{\Delta} + \lambda)(\mathring{\delta}(\boldsymbol{\varsigma}) - 2\eta) = 0.$$

Hence, condition (C1) implies that $\eta = \frac{1}{2}\mathring{\delta}(\boldsymbol{\varsigma})$. Substituting back into (5.2.6a),

$$0 = \mathring{\delta} \circ \mathring{L}(\boldsymbol{\varsigma})_i - \tfrac{2}{3}(d\eta)_i = \mathring{\Delta}\varsigma_i + \mathring{r}_i{}^j\varsigma_j + \tfrac{1}{3}\mathring{D}_i\mathring{\delta}(\boldsymbol{\varsigma}) - \tfrac{1}{3}\mathring{D}_i\mathring{\delta}(\boldsymbol{\varsigma}) = -\mathring{\Delta}_Y\varsigma_i$$

and hence condition (C2) implies $\varsigma_i = 0$, and hence $\eta = \frac{1}{2}\mathring{\delta}(\boldsymbol{\varsigma}) = 0$. Finally, substituting into (5.2.7b) we find that $\mathring{P}_L\bar{\eta}_{ij} = 0$ and hence condition (C3) implies $\bar{\eta}_{ij} = 0$. Collecting together the above, we see that $\eta_{ij} = 0$, $\varsigma_i = 0$, so $(D_v\widetilde{\Psi}_2)^*$ is injective and so $D_v\widetilde{\Psi}_2$ is surjective by the Fredholm alternative. □

Remark 31. Condition (C3) is also necessary for surjectivity of $D_v\widetilde{\Psi}$: given $\bar{\eta}_{ij} \in \ker\, \mathring{P}_L|_{\mathscr{S}^2_0(\mathcal{S},\mathring{\boldsymbol{h}}))}$, clearly

$$(D_v\widetilde{\Psi})^*(\mathbf{0},\mathbf{0},\mathbf{0},\bar{\boldsymbol{\eta}}) = 0.$$

Moreover, $\ker\, \mathring{\mathcal{P}}^{(1)} \subset \ker\, (D_v\widetilde{\Psi})^*_1$ and so injectivity of $\mathring{\mathcal{P}}^{(1)}$ is also necessary for surjectivity of $D_v\widetilde{\Psi}$. In particular, tracefree Codazzi tensors arise once more as obstructions. On the other hand, $\ker\, \mathring{\mathcal{P}}^{(0)} \subset \ker\, (D_v\widetilde{\Psi})_1$ and so injectivity of $\mathring{\mathcal{P}}^{(0)}$ is necessary for injectivity of $D_v\widetilde{\Psi}$ —in particular, given a conformal Killing field $\bar{\xi}_i$ and a tracefree Codazzi tensor χ_{ij},

$$(D_v\widetilde{\Psi})_1(\boldsymbol{\chi},\bar{\boldsymbol{\xi}}) = 0.$$

Whether or not conformal Killing vectors enter into the kernel $(D_v\widetilde{\Psi})^*_1$ is not so clear. The necessity of conditions (C1) and (C2) is also not clear.

Remark 32. In the above, we could have instead used the vanishing of the index to establish surjectivity. Recall that the Atiyah–Singer index theorem (see [61], for example) relates the analytical and topological index of an elliptic operator over a compact manifold. For an odd-dimensional base manifold $\mathcal{S}$ the topological index vanishes —see the discussion in [61]— and so the index theorem guarantees that an injective elliptic operator defined over an odd-dimensional manifold must in fact be an isomorphism of the appropriate Banach spaces.

5.2.3 The sufficiency argument

The previous section establishes the existence of perturbative solutions of the auxiliary system of equations $\widetilde{\Psi} = 0$, or in our terminology, *candidate solutions* of the ECEs. We must still show that our candidate solutions indeed solve $\Psi = 0$ —that the auxiliary equations are *sufficient*. There are two potential sources for the introduction of spurious solutions —that is to say, solutions of $\widetilde{\Psi} = 0$ which fail to be solutions of the ECEs— namely, (1): the composition of the operator $\mathring{\mathcal{D}}^*$ on the equation $J_{ijk} = 0$, and (2): the gauge reduction. We will see that the fact there are no spurious solutions follows essentially from the integrability conditions (4.1.11a)–(4.1.11b), as in the previous chapter.

Remark 33. Issue (2) is essentially the same as that arising in the procedure of hyperbolic reduction of the Einstein field equations. We will show that $Q^{\boldsymbol{X}}_i = 0$ for any sufficiently small perturbation of the background solution, given $\mathring{Q}^{\boldsymbol{X}}_i = 0$ (which is satisfied by construction). This is similar in spirit to a "gauge propagation" argument in the context of the Cauchy problem —there, however, one assumes that the gauge condition is satisfied on some initial hypersurface, and then shows that it is "propagated" in time.

We begin with a technical lemma concerning the behaviour of kernels of families of elliptic operators.

Proposition 12. Let $\mathcal{Q}_{\boldsymbol{h}} : E \longrightarrow F$ denote a family of linear second-order elliptic operators between bundles E, F over a closed manifold $\mathcal{S}$, depending on the choice of Riemannian metric, $\boldsymbol{h}$, and given in terms of some local coordinate system by

$$\mathcal{Q}_{\boldsymbol{h}}(\boldsymbol{\eta}) = \sum_{\alpha,\beta=1}^{3} \boldsymbol{a}_0(\boldsymbol{h})^{\alpha\beta} \cdot \partial_\alpha\partial_\beta\boldsymbol{\eta} + \boldsymbol{a}_1(\boldsymbol{h})^{\alpha} \cdot \partial_\alpha\boldsymbol{\eta} + \boldsymbol{a}_2(\boldsymbol{h}) \cdot \boldsymbol{\eta},$$

where, for each α, β, $\boldsymbol{a}_0(\boldsymbol{h})^{\alpha\beta}, \boldsymbol{a}_1(\boldsymbol{h})^{\alpha}, \boldsymbol{a}_2(\boldsymbol{h}) \in \mathrm{End}(E, F)$. Suppose moreover that for each α, β the maps

$$\begin{array}{rccc} M_I : & H^2(\mathscr{S}^2(\mathcal{S})) & \longrightarrow & B(H^I(E), L^2(F)) \\ & \boldsymbol{h} & \longmapsto & \boldsymbol{a}_I(\boldsymbol{h}) \end{array}$$

(indexed by $I = 0, 1, 2$) are Lipschitz continuous at $\boldsymbol{h} = \mathring{\boldsymbol{h}}$. Then, if $\mathcal{Q}_{\mathring{\boldsymbol{h}}}$ is injective on $H^2(E)$ then there exists some $\varepsilon > 0$ such that $\mathcal{Q}_{\boldsymbol{h}}$ is also injective on $H^2(E)$ for $\boldsymbol{h}$ satisfying $\|\boldsymbol{h} - \mathring{\boldsymbol{h}}\|_{H^4} < \varepsilon$.

The special case $\mathcal{Q}_{\boldsymbol{h}} = \mathcal{K}^*_{\boldsymbol{h}} \circ \mathcal{K}_{\boldsymbol{h}}$ of the above was given in Proposition 8 of the previous chapter; Proposition 8 can be subsumed into the above. Since the proof works in the same way for any second-order elliptic operator satisfying the given assumptions, we will not repeat the details here.

The proof of the following closely follows that of Proposition 9 in Section 4.3.3, the only addition being that we will require a second application of Proposition 12, this time with

$$\mathcal{Q} \equiv \Delta_Y,$$

which clearly satisfies the required assumptions. Defining the map

$$\omega : (\phi, \bar{\boldsymbol{T}}; \boldsymbol{T}, \boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}) \mapsto \begin{pmatrix} \chi_{ij} + \frac{1}{3}\phi\mathring{h}_{ij} \\ \bar{\boldsymbol{S}}(\bar{\boldsymbol{X}}, \bar{\boldsymbol{T}}) \\ \boldsymbol{S}(\boldsymbol{X}, \boldsymbol{T}) \end{pmatrix},$$

as in Chapter 4, we have the following:

Proposition 13. There exists an open neighbourhood of the trivial data $(\phi, \bar{\boldsymbol{T}}, \boldsymbol{T}) = (0, \boldsymbol{0}, \boldsymbol{0})$, $\mathcal{U}' \subset \mathcal{U}$ such that for each $u \in \mathcal{U}'$ the corresponding candidate solution $w(u) = w(u; \nu(u))$ solves the ECEs. Here, $\mathcal{U}$ and ν are as given in Proposition 11.

Proof. Since the auxiliary equations are satisfied, (in particular $\Lambda_i = \bar{\Lambda}_i = 0$) we have

$$\mathring{\mathcal{D}}^*(\boldsymbol{J}) = 0, \tag{5.2.8a}$$

$$\epsilon_{ijk} D^k J^{ij}{}_l = 0, \tag{5.2.8b}$$

$$\Delta_Y(Q^{\boldsymbol{X}})_j = -K_{ik}J_j{}^{ik} + K_{jk}J^{ik}{}_i + KJ_j{}^i{}_i. \tag{5.2.8c}$$

We aim then to show that $J_{ijk} = 0$ and $Q^{\boldsymbol{X}}_i = 0$ (implying that $V_{ij} = 0$). Let us first show that $J_{ijk} = 0$. Collectively, (5.2.8a)–(5.2.8b) are precisely

$$\mathcal{K}_{\boldsymbol{h}}(\boldsymbol{J}) = 0.$$

Recall that $\mathring{\mathcal{K}}(\boldsymbol{J}) = 0$ is equivalent to $\mathring{\mathcal{K}}^{(0)}(\boldsymbol{J}) = 0$ —see Remark 27. Hence, condition (C4) implies that $\mathring{\mathcal{K}}$ is injective. Applying Proposition 12 to $\mathcal{Q}_{\boldsymbol{h}} \equiv \mathcal{K}^* \circ \mathcal{K}$, we find that $\mathcal{Q}_{\boldsymbol{h}}$ is also injective for sufficiently small metric perturbations, from which it follows by integration by parts that $\mathcal{K}_{\boldsymbol{h}}$ is also injective. Hence, we find that $J_{ijk} = 0$. [2]

[2] This is the same argument as in Proposition 8 of Chapter 4.

Substituting $J_{ijk} = 0$ into (5.2.8c) we have

$$\Delta_Y (Q^{\boldsymbol{X}})_j = 0.$$

Now, $\mathring{\Delta}_Y$ is injective by condition (C2). Hence, applying Lemma 12 to $\mathcal{Q} \equiv \Delta_Y$, we see that Δ_Y is injective for sufficiently-small metric perturbations, implying that $Q_i^{\boldsymbol{X}} = 0$.

Making $\mathcal{U}'$ sufficiently small guarantees (by continuity) that the candidate solution $w(u)$ is a small perturbation of the background data set; in particular, that the metric $\boldsymbol{h}$ is a small perturbation of $\mathring{\boldsymbol{h}}$. Hence, for sufficiently small $\mathcal{U}'$, the system (5.2.8a)–(5.2.8c) admits only the trivial solution, $J_{ijk} = 0$, $Q_i^{\boldsymbol{X}} = 0$, and it follows that $w(u)$, for $u \in \mathcal{U}'$, necessarily satisfies the ECEs. □

5.2.4 The main theorem

In addition to proving the existence of solutions to the ECEs, we would also like to know whether two distinct choices for the freely-prescribed field give rise to distinct solutions of the ECEs —in other words, whether the map from free data to the space of solutions, furnished by the IFT and denoted here by ν, is injective. To do so, we need only show that $D_u\widetilde{\Psi}$ is injective at the background solution.

We are now in a position to prove the main result of this section:

Theorem 3. Let $(\mathcal{S}, \mathring{\boldsymbol{h}})$ be a smooth closed Riemannian manifold with constant scalar curvature $\mathring{r} = 2\lambda$, and satisfying conditions (C1)–(C4). Then, for $s \geq 4$, there exists an open neighbourhood $\mathcal{U} \subset \mathcal{Y}^s$ of $(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$, an open neighbourhood $\mathcal{W} \subset \mathcal{X}^s$ of $(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \mathring{\boldsymbol{h}})$ and a smooth map $\nu : \mathcal{U} \to \mathcal{W}$ such that, defining

$$u \equiv (\phi, \bar{\boldsymbol{T}}, \boldsymbol{T}), \qquad \nu(u) \equiv \big(\boldsymbol{\chi}(u), \bar{\boldsymbol{X}}(u), \boldsymbol{X}(u), \boldsymbol{h}(u)\big),$$

the following assertions hold:

i) for each $(\phi, \bar{\boldsymbol{T}}, \boldsymbol{T}) \in \mathcal{U}$,

$$w(u) \equiv \big(\boldsymbol{\chi}(u) + \tfrac{1}{3}\phi\mathring{\boldsymbol{h}},\ \bar{\boldsymbol{S}}(\bar{\boldsymbol{X}}(u), \bar{\boldsymbol{T}}),\ \boldsymbol{S}(\boldsymbol{X}(u), \boldsymbol{T}),\ \boldsymbol{h}(u)\big)$$

is a solution to the ECEs with cosmological constant λ;

ii) the map $u \mapsto w(u)$ is injective if we restrict to free datum ϕ to the sub-Banach space, $\bar{H}^{s-1}(\mathscr{C}(\mathcal{S}))$, consisting of functions which integrate to zero over $\mathcal{S}$.

Proof. Collecting together Propositions 11 and 13 we obtain (i). For (ii), first recall from (5.1.16) that

$$D_u\widetilde{\Psi}(\breve{\phi}, \breve{\bar{\boldsymbol{T}}}, \breve{\boldsymbol{T}}) = \begin{pmatrix} \mathring{\mathcal{R}}(\breve{\bar{\boldsymbol{T}}}) - \frac{1}{6}\mathring{L}(d\phi)_{jk} \\ \mathring{\delta}(\breve{\bar{\boldsymbol{T}}})_i \\ \mathring{\delta}(\breve{\boldsymbol{T}})_i \\ -\breve{T}_{ij} \end{pmatrix}.$$

It follows immediately from the last row that $\breve{T}_{ij} = 0$, leaving only the first two rows, which are equivalent to

$$\mathring{\mathcal{P}}^{(0)} \begin{pmatrix} \breve{\bar{T}}_{jk} \\ -\frac{1}{3}\mathring{D}_i\breve{\phi} \end{pmatrix} = 0.$$

Now, by condition (C4), $\mathring{\mathcal{P}}^{(0)}$ is injective, and hence $\breve{\bar{T}}_{ij} = 0$ and $d\breve{\phi} = 0$, hence $\breve{\phi}$ is constant. Restricting $\breve{\phi}$ to the Banach space $\bar{H}^{s-1}(\mathscr{C}(\mathcal{S}))$ we find that $\breve{\phi} = 0$. □

Remark 34. It is not clear whether conditions (C1)–(C4) are *necessary* for the existence of non-linear perturbative solutions of the ECEs. It is clear that if one or more of $\mathring{\Delta}_Y, \mathring{P}_L, (\mathring{\Delta}+\lambda), \mathring{\mathcal{P}}^{(0)}, \mathring{\mathcal{P}}^{(1)}$ fails to be injective, then $D_v\widetilde{\Psi}$ will fail to be invertible. However, the use of a different auxiliary constraint map would inevitably lead to a different set of conditions —c.f. Chapter 6.

5.3 Exploring the conditions

The purpose of this section is to discuss some particular situations in which the conditions (C1)–(C4) are satisfied. We will see, in particular, that if the Ricci curvature of $\mathring{h}_{ij}$ is suitably "pinched" in the appropriate way then conditions (C1), (C2) and (C3) are necessarily satisfied. Here, when we say that the curvature is *pinched* we mean that the sectional curvatures —see A.3 of the Appendix— $\sigma_I,\ I = 1, 2, 3$ at each $p \in \mathcal{S}$ (or equivalently, the eigenvalues of the Ricci curvature tensor) are close to one another so that $(\mathcal{S}, \mathring{\boldsymbol{h}})$ is, in a sense, close to being Einstein or equivalently (since we consider dim $\mathcal{S} = 3$) close to being a space form.

5.3.1 Notions of curvature pinching

There are various notions of pointwise pinching in the literature. One such notion, with applications to Ricci flow (see [70]), is the following

$$\varrho = \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{(\sigma_1 + \sigma_2 + \sigma_3)^2},$$

for which increased pinching is characterised by $\varrho \to 0$, and $\varrho = 0$ precisely when all sectional curvatures are all equal. When considering positive sectional curvature, another commonly used notion is

$$\delta = \min_{I,J} \frac{\sigma_I}{\sigma_J}$$

as in the Differentiable Sphere Theorem [71], for instance, for which increased pinching corresponds to $\delta \to 1$, and now equality of the sectional curvatures is characterised by $\delta = 1$. If $0 < \alpha_1 \le \sigma_I \le \alpha_2$, then it straightforward to see that

$$3\left(\frac{\alpha_1}{\alpha_2}\right)^2 \varrho \le |1-\delta|^2 \le 9\left(\frac{\alpha_2}{\alpha_1}\right)^2 \varrho,$$

and so the two notions are equivalent. A similar conclusion holds for negative sectional curvature, in which case the pinching coefficient is defined as

$$\delta = -\min_{I,J} \frac{\sigma_I}{\sigma_J}.$$

Here, since we are concerned with constant scalar curvature metrics ($r = 2\lambda$), there are two independent sectional curvatures at each point. A convenient parametrisation is

$$\sigma_\pm = -\tfrac{2}{\sqrt{3}}\lambda + \tfrac{1}{2}(\pm 1 + \sqrt{3})\sigma_1 + \tfrac{1}{2}(\mp 1 + \sqrt{3})\sigma_2,$$

in terms of which, for $\lambda \neq 0$, $\varrho = (1/4\lambda^2)(\sigma_-^2 + \sigma_+^2)$; ϱ can be thought of as the Euclidean norm on $\mathbb{R}^2$. Note, in particular, that if ϱ is sufficiently small then all sectional curvatures have the same sign as λ. In this case, the eigenvalues (at each $p \in \mathcal{S}$) of the Ricci tensor (thought of now as a linear map from $T\mathcal{S}$ to $\Lambda^1(\mathcal{S})$), are given by

$$\tfrac{1}{2}(\sigma_2+\sigma_3-\sigma_1), \qquad \tfrac{1}{2}(\sigma_3+\sigma_1-\sigma_2), \qquad \tfrac{1}{2}(\sigma_1+\sigma_2-\sigma_3)$$

—see A.3 of the Appendix— and also have the same sign as λ. Therefore sufficient pinching implies positive or negative-definiteness of the Ricci curvature, for $\lambda > 0$ and $\lambda < 0$, respectively. From now on, a constant scalar curvature metric will be said to have *pinched* curvature if (σ_-, σ_+) is small with respect to the norm ϱ (or any equivalent norm on $\mathbb{R}^2$), at each point $p \in \mathcal{S}$, so that the sectional curvatures $\sigma_1, \sigma_2, \sigma_3$ are close to being equal to one another (and therefore close to $2\lambda/3$). When $\lambda > 0$ we will talk of *positive pinching* and when $\lambda < 0$ we will talk of *negative pinching.* The results given here will be qualitative; quantifying the necessary degree of pinching of course requires a choice of norm.

5.3.2 Conditions (C1) – (C3)

It is clear that condition (C1) trivialises for $\lambda < 0$ as a result of positive semi-definiteness of $-\mathring{\Delta}$. Similarly, condition (C2) trivialises for background metrics with negative-definite Ricci curvature (in particular for hyperbolic metrics) since then $\mathring{\Delta}_Y > 0$. Here, by $\mathring{\Delta}_Y > 0$ (more generally $\mathcal{K} > 0$ for any self-adjoint elliptic operator $\mathcal{K}$) we mean that

$$\int_{\mathcal{S}} \langle \mathring{\Delta}_Y(\boldsymbol{V}), \boldsymbol{V}\rangle_{\mathring{h}}\, d\mathring{\mu} > 0$$

for all non-zero $\boldsymbol{V} \in \Gamma(\Lambda^1(\mathcal{S}))$. Equivalently, since $\mathcal{S}$ is closed, the *Spectral Theorem* —see Theorem 5.8 of [43]— implies that $\mathring{\Delta}_Y > 0$ if and only if the eigenvalues of $\mathring{\Delta}_Y$ are strictly positive.

For what follows, define $\mathfrak{R} : \mathscr{S}^2(\mathcal{S}) \to \mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$ as

$$\mathfrak{R}(\boldsymbol{\eta})_{ij} \equiv 2r_{(i}{}^{l}\eta_{j)l} - 2r_{ikjl}\eta^{kl} = 6r_{(i}{}^{k}\eta_{j)k} - r\eta_{ij},$$

where the equality follows from the Kulkarni–Nomizu decomposition of the Riemann tensor (see Section 2.1). In particular then,

$$\langle \boldsymbol{\eta}, P_L\boldsymbol{\eta}\rangle = \|D\boldsymbol{\eta}\|^2 + \langle \boldsymbol{\eta}, \mathfrak{R}(\boldsymbol{\eta})\rangle - \tfrac{2}{3}r\|\boldsymbol{\eta}\|^2. \tag{5.3.1}$$

It is easily computed that for an Einstein metric, $\mathfrak{R}(\boldsymbol{\eta})_{ij} = r\eta_{ij}$, and so in the elliptic case ($r = 2\lambda = 6$)

$$\langle \boldsymbol{\eta}, \mathfrak{R}(\boldsymbol{\eta})\rangle - \tfrac{2}{3}r\|\boldsymbol{\eta}\|^2 = \tfrac{1}{3}r\|\boldsymbol{\eta}\|^2 = 2\|\boldsymbol{\eta}\|^2,$$

implying that $P_L > 0$ on $\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$. It follows immediately that if h_{ij} has sufficiently positively-pinched curvature then the same conclusion holds and condition (C3) is satisfied. An analogous result for negatively-pinched curvature also holds:

Proposition 14. The operator P_L satisfies the following *Bochner formula*

$$\langle \boldsymbol{\eta}, P_L\boldsymbol{\eta}\rangle = \|\mathcal{D}\boldsymbol{\eta}\|^2 + 2\|\delta(\boldsymbol{\eta})\|^2 + \tfrac{1}{2}\langle \boldsymbol{\eta}, \mathfrak{R}(\boldsymbol{\eta})\rangle - \tfrac{2}{3}r\|\boldsymbol{\eta}\|^2. \tag{5.3.2a}$$

It follows that $P_L > 0$ on $\mathscr{S}^2_0(\mathcal{S},\boldsymbol{h})$ for a metric with sufficiently negatively-pinched curvature. Such a metric satisfies (C3).

Proof. From direct computation,

$$\mathcal{D}^* \circ \mathcal{D}(\boldsymbol{\eta})_{ij} = P_L \eta_{ij} + \tfrac{2}{3} r \eta_{ij} + L \circ \delta(\boldsymbol{\eta})_{ij} - \tfrac{1}{2}\mathfrak{R}(\boldsymbol{\eta})_{ij}$$

Integrating by parts against η^{ij} and rearranging,

$$\int_{\mathcal{S}} \langle \boldsymbol{\eta}, P_L(\boldsymbol{\eta})\rangle \, d\mu_{\boldsymbol{h}} = \int_{\mathcal{S}} \left(\|\mathcal{D}(\boldsymbol{\eta})\|^2 + 2\|\delta(\boldsymbol{\eta})\|^2 - \tfrac{2}{3} r \|\boldsymbol{\eta}\|^2 + \tfrac{1}{2}\langle \boldsymbol{\eta}, \mathfrak{R}(\boldsymbol{\eta})\rangle \right) \, d\mu_{\boldsymbol{h}}. \tag{5.3.3}$$

Evaluating the algebraic terms for a hyperbolic metric ($r = 2\lambda = -6$), one obtains

$$-\tfrac{2}{3} r \|\boldsymbol{\eta}\|^2 + \tfrac{1}{2}\langle \boldsymbol{\eta}, \mathfrak{R}(\boldsymbol{\eta})\rangle = -\tfrac{1}{6} r \|\boldsymbol{\eta}\|^2 = \|\boldsymbol{\eta}\|^2 \geq 0,$$

with equality if and only $\eta_{ij} = 0$. In other words, the curvature terms in (5.3.3) are positive-definite for a hyperbolic metric. The same conclusion clearly holds for metric which are sufficiently negatively-pinched. □

Returning to the positive scalar curvature case, define

$$(D_{\text{sym}} \boldsymbol{\eta})_{ijk} \equiv D_i \eta_{jk} + D_j \eta_{ki} + D_k \eta_{ij},$$

for which the following Bochner formula holds

$$\langle \boldsymbol{\eta}, P_L \boldsymbol{\eta}\rangle = \tfrac{1}{2}\|D_{\text{sym}} \boldsymbol{\eta}\|^2 - 2\|\delta(\boldsymbol{\eta})\|^2 + 2\langle \boldsymbol{\eta}, \mathfrak{R}(\boldsymbol{\eta})\rangle - \tfrac{2}{3} r \|\boldsymbol{\eta}\|^2. \tag{5.3.4}$$

For an elliptic manifold,

$$2\langle \boldsymbol{\eta}, \mathfrak{R}(\boldsymbol{\eta})\rangle - \tfrac{2}{3} r \|\boldsymbol{\eta}\|^2 = 8\|\boldsymbol{\eta}\|^2.$$

If P_L is restricted to act on the space $\mathscr{S}_{TT}(\mathcal{S}; \boldsymbol{h}) \subset \mathscr{S}^2_0(\mathcal{S}; \boldsymbol{h})$, then one requires less pinching to establish $P_L > 0$ than via identity (5.3.1). Of course, in order to make use of (5.3.4) one first has to weaken (C3) to the assumption that P_L is injective on $\mathscr{S}_{TT}(\mathcal{S}; \boldsymbol{h})$. This will be explored elsewhere. The above discussion is related to well-known results on Einstein deformations, originally due to Koiso and Bourguignon —see [72] for a summary.

5.3.3 Condition (C4)

Recall that to satisfy condition (C4) we require that $\mathring{\boldsymbol{h}}$ admit neither global conformal Killing vectors nor tracefree Codazzi tensors. The difficulty in ensuring injectivity of $\mathcal{P}^{(0)}$ stems from the fact that the natural geometric conditions which guarantee non-existence of conformal Killing vectors and tracefree Codazzi tensors (both of which must be excluded if $\mathcal{P}^{(0)}$ is to be injective) are negativity and positivity conditions, respectively, on the Ricci curvature. It therefore appears difficult to establish both non-existence of conformal Killing vectors and tracefree Codazzi tensors simply by considerations of the curvature. One might hope that if one assumes the non-existence of either conformal Killing vectors or of tracefree Codazzi tensors, then the non-existence of the other may be established given appropriate restrictions of the curvature. However, certain curvature terms coupling X_i and Y_{ij} seem to hinder such an approach: indeed, suppose $(\boldsymbol{Y}, \boldsymbol{X})$ satisfies

$\mathcal{P}^{(0)}(\boldsymbol{Y}, \boldsymbol{X}) = 0$, then we have

$$\begin{aligned} 0 &= \delta(L(\boldsymbol{X}) + 2\mathcal{R}(\boldsymbol{Y}))_i \\ &= \delta \circ L(\boldsymbol{X})_i + 2\delta \circ \mathcal{R}(\boldsymbol{Y})_i \\ &= \delta \circ L(\boldsymbol{X})_i + \mathrm{curl} \circ \delta(\boldsymbol{Y})_i - 2\epsilon_{iml} r_j{}^l Y^{jm} \\ &= \delta \circ L(\boldsymbol{X})_i - 2\epsilon_{iml} r_j{}^l Y^{jm}, \end{aligned}$$

where the third line uses the identity

$$\delta \circ \mathcal{R}(\boldsymbol{Y})_i = \tfrac{1}{2}\mathrm{curl} \circ \delta(\boldsymbol{Y})_i - \epsilon_{iml} r_j{}^l Y^{jm},$$

and the fourth uses the second component of $\mathcal{P}^{(0)}(\boldsymbol{Y}, \boldsymbol{X}) = 0$. The curvature term $\epsilon_{iml} r_j{}^l Y^{jm}$ prevents us from decoupling X_i and Y_{ij}. If $\epsilon_{iml} r_j{}^l Y^{jm} = 0$, as in the case of an Einstein manifold, then one is left with the equation $\delta \circ L(\boldsymbol{X})_i = 0$, implying $X_i = 0$ if we assume the non-existence of conformal Killing vectors. Following this approach, one can prove the following:

Proposition 15. Suppose $(\mathcal{S}, \boldsymbol{h})$ is a closed Einstein manifold with (constant) scalar curvature $r = 2\lambda$, normalised to $\lambda = -3, 0, 3$. Then,

(i) For $\lambda = -3$ and $\alpha \geq 0$, $\ker \mathcal{P}^{(\alpha)} = \mathbf{C} \oplus \{\mathbf{0}\}$,

(ii) For $\lambda = 0$,

  (a) $\ker \mathcal{P}^{(0)} = \mathbf{C} \oplus \mathbf{c}$,

  (b) $\ker \mathcal{P}^{(\alpha)} = \mathbf{C} \oplus \mathbf{p}$ for $\alpha > 0$,

(iii) For $\lambda = 3$,

  (a) $\ker \mathcal{P}^{(0)} = \{\mathbf{0}\} \oplus \mathbf{c}$,

  (b) $\ker \mathcal{P}^{(\alpha)} = \{\mathbf{0}\} \oplus \{\mathbf{0}\}$ for $\alpha > 0$, unless $(\mathcal{S}, \boldsymbol{h})$ is isometric to $\mathbb{S}^3$ whereupon

$$\ker \mathcal{P}^{(\alpha)} = \{\mathbf{0}\} \oplus \mathrm{sp}\langle dx_I, I = 1, 2, 3, 4\rangle,$$

  where $x_I,\ I = 1, 2, 3, 4$, are the restriction to $\mathbb{S}^3 \subset \mathbb{R}^4$ of the standard Cartesian coordinate functions on $\mathbb{R}^4$.

Here we are using $\mathbf{C}$, $\mathbf{c}$ and $\mathbf{p}$ as shorthands for the linear spaces of tracefree Codazzi tensors, conformal Killing vectors and parallel covectors (covectors satisfying $D_i V_j = 0$). The proof is given in Appendix A.4.

Remark 35. Note, in particular, that a conformally rigid hyperbolic metric has $\ker \mathcal{P}^{(\alpha)} = \{\mathbf{0}\}$ so that condition (C4) is satisfied. The discussion of the previous section establishes that conditions (C1)–(C3) are also satisfied, and so the time symmetric case of Theorem 2 is recovered as a corollary of Theorem 3.

On the other hand, elliptic manifolds ($\lambda = 3$) fail to satisfy condition (C4) for the same reason that they fail condition (C2), namely that the existence of non-trivial Killing vector fields imply a non-trivial kernel. For $\mathbb{S}^3$ (i.e. initial data for the de Sitter spacetime), the operator $\mathcal{P}^{(1)}$ also has a non-trivial kernel. It is interesting to note the appearance of the *static potentials* x_I in this case.

In order to go beyond Einstein metrics, we see that we need to gain control over the coupling term $\epsilon_{iml}r_j{}^l Y^{jm}$. For the remainder of this section we will restrict to background metrics of positive scalar curvature. Here we will consider the effect of strengthening condition (C2) to the following

$$(C2)': \quad \Delta_Y > 0 \quad \text{as a map} \quad \Gamma(\Lambda^1(\mathcal{S})) \to \Gamma(\Lambda^1(\mathcal{S})).$$

Since this implies $\|D\boldsymbol{X}\|^2_{L^2} \geq |\mathrm{Ric}(\boldsymbol{X},\boldsymbol{X})|^2_{L^2}$ for all covector fields X_i, this will allow us to trade off $\|D\boldsymbol{X}\|^2$ terms for $\mathrm{Ric}(\boldsymbol{X},\boldsymbol{X})$ terms and hence compensate for the $\epsilon_{iml}r_j{}^l Y^{jm}$ term, which is of indefinite sign. Recall that a prerequisite for condition (C4) is the non-existence of conformal Killing fields. If (C2) is strengthened to (C2)$'$, then one finds that indeed there can be no conformal Killing vector fields. To see this, note that for an arbitrary covector field, X_i, one has

$$\int_{\mathcal{S}} \langle \boldsymbol{X}, \Delta_Y \boldsymbol{X}\rangle \; d\mu_h = \int_{\mathcal{S}} \tfrac{1}{2}\|\mathcal{L}_{\boldsymbol{X}}\boldsymbol{h}\|^2 - |\delta(\boldsymbol{X})|^2 \; d\mu_h.$$

If X_i a conformal Killing (co)vector, one obtains

$$\int_{\mathcal{S}} \langle \boldsymbol{X}, \Delta_Y \boldsymbol{X}\rangle \; d\mu_h = -\tfrac{1}{3}\int_{\mathcal{S}} |\delta(\boldsymbol{X})|^2 \; d\mu_h$$

and it follows from $\Delta_Y > 0$ that in fact $\delta(\boldsymbol{X}) = 0$, so that X_i is a Killing vector field and hence that $X_i = 0$, by injectivity of Δ_Y. In light of this, it is plausible that when combined with a positive-pinching condition (required to eliminate the possibility of tracefree Codazzi tensors), condition (C2)$'$ forces injectivity of $\mathcal{P}^{(\alpha)}$. That this is indeed the case is shown in the following two subsections.

Remark 36. Condition (C2)' has a natural geometric interpretation in terms of the stability of the Dirichlet energy for mappings of Riemannian manifolds; the operator $\mathring{\Delta}_Y$ arises as the *stability operator* —see [73] for more details.

Integral identities for solutions of $\mathcal{P}^{(\alpha)}(\boldsymbol{Y},\boldsymbol{X}) = 0$

We show that if $(\boldsymbol{X},\boldsymbol{Y})$ satisfies $\mathcal{P}^{(\alpha)}(\boldsymbol{X},\boldsymbol{Y}) = 0$, then it satisfies an identity of the form

$$0 = \int_{\mathcal{S}} \left(\|D\boldsymbol{Y}\|^2 + 2\|D\boldsymbol{X}\|^2 + \mathscr{R}(\boldsymbol{X},\boldsymbol{Y})\right) \; d\mu_h, \tag{5.3.5}$$

where $\mathscr{R}(\boldsymbol{X},\boldsymbol{Y})$ is an algebraic expression that is quadratic in $(\boldsymbol{X},\boldsymbol{Y})$.

Lemma 13. A solution of $\mathcal{P}_{(\alpha)}(\boldsymbol{Y},\boldsymbol{X}) = 0$ satisfies the following integral identities:

$$0 = \int_{\mathcal{S}} \Big(\|D\boldsymbol{Y}\|^2 - \tfrac{1}{2}(3\alpha-1)\|D\boldsymbol{X}\|^2 + \tfrac{1}{2}(3\alpha-1)|\delta(\boldsymbol{X})|^2 + \mathscr{R}^{(\alpha)}_1(\boldsymbol{X},\boldsymbol{Y})\Big) \; d\mu_h, \tag{5.3.6a}$$

$$0 = \int_{\mathcal{S}} \Big(\tfrac{1}{2}(1+\alpha)\|D\boldsymbol{X}\|^2 + \tfrac{1}{6}(1-3\alpha)|\delta(\boldsymbol{X})|^2 + \mathscr{R}^{(\alpha)}_2(\boldsymbol{X},\boldsymbol{Y})\Big) \; d\mu_h, \tag{5.3.6b}$$

where $\mathscr{R}^{(\alpha)}_1(\cdot,\cdot)$ and $\mathcal{R}^{(\alpha)}_2(\cdot,\cdot)$ denote the following quadratic forms

$$\mathscr{R}^{(\alpha)}_1(\boldsymbol{X},\boldsymbol{Y}) \equiv \tfrac{1}{2}(1-3\alpha)r_{ij}X^iX^j + \tfrac{1}{2}\mathfrak{R}(\boldsymbol{Y})_{ij}Y^{ij} - \tfrac{1}{2}\epsilon_j{}^{nk}r_{imnk}Y^{jm}X^i, \tag{5.3.7a}$$

$$\mathscr{R}^{(\alpha)}_2(\boldsymbol{X},\boldsymbol{Y}) \equiv -\tfrac{1}{2}(1-\alpha)r_{ij}X^iX^j + \epsilon_{imp}r_j{}^pY^{jm}X^i. \tag{5.3.7b}$$

Proof. We generalise the derivation of identities (4.1.15a)–(4.1.15b) of Section 4.1.3. Recall that, by virtue of Y_{ij} being tracefree,

$$\epsilon^l{}_{ij}Y_{kl} + \epsilon^l{}_{jk}Y_{il} + \epsilon^l{}_{ki}Y_{jl} = 0.$$

Taking the divergence and substituting for the $\delta(\boldsymbol{Y})-$term using the second component of $\mathcal{P}^{(\alpha)}(\boldsymbol{Y},\boldsymbol{X}) = 0$, we obtain

$$\epsilon_{kjl}D^lY_i{}^j = -\alpha D_iX_k + \alpha D_kX_i + \epsilon_{ijl}D^lY_k{}^j.$$

Substituting back into the first component of $\mathcal{P}^{(\alpha)}(\boldsymbol{Y},\boldsymbol{X}) = 0$,

$$D_nY_{im} - D_mY_{in} - \tfrac{1}{2}\epsilon_{mnj}D_iX^j + \tfrac{1}{2}\alpha\epsilon_{mnj}D_iX^j + \tfrac{1}{3}\epsilon_{imn}D_jX^j - \tfrac{1}{2}\epsilon_{mnj}D^jX_i - \tfrac{1}{2}\alpha\epsilon_{mnj}D^jX_i = 0.$$

Taking another contracted derivative, commuting derivatives on the second term and substituting again for $\delta(\boldsymbol{Y})$, one obtains the following

$$\begin{aligned} D_jD^jY_{im} - \alpha\epsilon_{ijn}D_mD^nX^j + \tfrac{1}{2}\epsilon_{mjn}D^nD_iX^j - \tfrac{1}{2}\alpha\epsilon_{mjn}D^nD_iX^j + \tfrac{1}{3}\epsilon_{imn}D^nD_jX^j \\ - r_{mj}Y_i^j + r_{ijmn}Y^{jn} - \tfrac{1}{4}\epsilon_m{}^{nk}r_{ijnk}X^j - \tfrac{1}{4}\epsilon_m{}^{nk}\alpha X^jr_{ijnk} = 0. \end{aligned}$$

The latter decomposes into symmetric and antisymmetric parts to give the two following second-order equations for $Y_{ij},\ X_i$

$$\begin{aligned} \Delta Y_{im} - r_{j(m}Y_{i)}{}^j + r_{(i|j|m)n}Y^{jn} + \alpha\epsilon_{(i}{}^{jn}D_{m)}D_jX_n - \tfrac{1}{2}\epsilon_{(i}{}^{jn}D_{|j|}D_{m)}X_n \\ + \tfrac{1}{2}\alpha\epsilon_{(i}{}^{jn}D_{|j|}D_{m)}X_n - \tfrac{1}{4}\epsilon_{(i}{}^{nk}r_{m)jnk}X^j - \tfrac{1}{4}\epsilon_{(i}{}^{nk}r_{m)jnk}\alpha X^j = 0, \quad (5.3.8a) \\ (1+\alpha)\Delta X^p + \left(\tfrac{1}{3}-\alpha\right)D^pD_iX^i - 2Y^{ij}\epsilon^p{}_{jm}r_i{}^m + (1-\alpha)X^ir^p{}_i = 0. \quad (5.3.8b) \end{aligned}$$

To derive (5.3.6b), contract (5.3.8b) with X_p and integrate by parts. To derive (5.3.6a), first we calculate

$$\begin{aligned} \int_{\mathcal{S}} \epsilon_{mik}Y_j{}^kD^jD^iX^m\ d\mu_h &= \int_{\mathcal{S}} -\epsilon_{mik}D^jY_j{}^kD^iX^m\ d\mu_h \\ &= \int_{\mathcal{S}} \left(-D_iX_mD^iX^m + D^iX^mD_mX_i\right)\ d\mu_h \\ &= \int_{\mathcal{S}} \left(-\|D\boldsymbol{X}\|^2 + |\delta(\boldsymbol{X})|^2 - r_{ij}X^iX^j\right)\ d\mu_h, \qquad (5.3.9) \end{aligned}$$

where the second line follows using $\delta(\boldsymbol{Y})_i = -\alpha\mathrm{curl}(\boldsymbol{X})_i$ and the third line follows from the identity

$$\int_{\mathcal{S}} D_iX_jD^jX^i\ d\mu_h = \int_{\mathcal{S}} \left(\|D\boldsymbol{X}\|^2 - r_{ij}X^iX^j\right)\ d\mu_h,$$

which is derived by integrating by parts, commuting the resulting double-derivative and integrating by parts once more. Contracting (5.3.8a) with Y^{im}, integrating by parts and using (5.3.9), one obtains (5.3.6b).

$$\begin{aligned} \mathscr{R}_1^{(\alpha)}(\boldsymbol{X},\boldsymbol{Y}) \equiv{}& \tfrac{1}{2}(1-3\alpha)r_{ij}X^iX^j + r_{jm}Y_i^mY^{ij} - r_{imjn}Y^{ij}Y^{mn} \\ & - \tfrac{1}{4}(1+\alpha)\epsilon_j{}^{nk}r_{imnk}Y^{jm}X^i - \tfrac{1}{2}(1-\alpha)\epsilon_j{}^{nk}r_{inmk}Y^{jm}X^i \\ ={}& \tfrac{1}{2}(1-3\alpha)r_{ij}X^iX^j + r_{jm}Y_i^mY^{ij} - r_{imjn}Y^{ij}Y^{mn} - \tfrac{1}{2}\epsilon_j{}^{nk}r_{imnk}Y^{jm}X^i \\ ={}& \tfrac{1}{2}(1-3\alpha)r_{ij}X^iX^j + \tfrac{1}{2}\mathfrak{R}(\boldsymbol{Y})_{ij}Y^{ij} - \tfrac{1}{2}\epsilon_j{}^{nk}r_{imnk}Y^{jm}X^i, \end{aligned}$$

where the second line follows from the first Bianchi identity. □

Remark 37. Note that in the case $\alpha = -1$, the leading term of (5.3.8b) vanishes, consistent with the fact that $\mathcal{P}^{(-1)}$ is not elliptic —see Remark 26. Note also that, by setting $X_i = 0$ in (5.3.5), one recovers (4.1.17) of Section 4.1.4, which was used there to establish non-existence of constant-trace Codazzi tensors for manifolds of positive sectional curvature.

Adding three times (5.3.6b) to (5.3.6a), we obtain an identity of the form (5.3.5) with

$$\begin{aligned}\mathscr{R}(\boldsymbol{X},\boldsymbol{Y}) &\equiv \mathscr{R}_1^{(\alpha)}(\boldsymbol{X},\boldsymbol{Y}) + 3\mathscr{R}_2^{(\alpha)}(\boldsymbol{X},\boldsymbol{Y}) \\ &= -r_{ij}X^iX^j + \tfrac{1}{2}\mathfrak{R}(\boldsymbol{Y})_{ij}Y^{ij} - \tfrac{1}{2}\epsilon_j{}^{nk}r_{imnk}Y^{jm}X^i + 3\epsilon_{imp}r_j{}^pX^iY^{jm} \\ &= -r_{ij}X^iX^j + \tfrac{1}{2}\mathfrak{R}(\boldsymbol{Y})_{ij}Y^{ij} + 4\epsilon_{ikm}r_j{}^mX^iY^{jk},\end{aligned}$$

where the last line follows from the Kulkarni–Nomizu decomposition of the Riemann tensor. It is interesting to note that the identity does not depend on α. Note that, since we are restricting to background manifolds of positive section curvature, the first term of $\mathscr{R}(\boldsymbol{X},\boldsymbol{Y})$ is not positive-definite. We shall need to control this term in order to show injectivity of $\mathcal{P}^{(\alpha)}$.

Using condition (C2)′

Assuming the background metric $\mathring{\boldsymbol{h}}$ satisfies condition (C2)′, it follows immediately that

$$\int_{\mathcal{S}} \|\mathring{D}\boldsymbol{X}\|^2 \, d\mathring{\mu} \geq \int_{\mathcal{S}} \mathring{r}_{ij}X^iX^j \, d\mathring{\mu},$$

for all X_i. Hence, if $(\boldsymbol{Y},\boldsymbol{X})$ solves $\mathcal{P}^{(\alpha)}(\boldsymbol{Y},\boldsymbol{X})$ (for a given, fixed, $\alpha \in \mathbb{R}$), then from identity (5.3.5) we see that

$$\begin{aligned}0 &= \int_{\mathcal{S}} \Big(\|\mathring{D}\boldsymbol{Y}\|^2 + 2\|\mathring{D}\boldsymbol{X}\|^2 + \mathring{\mathscr{R}}(\boldsymbol{X},\boldsymbol{Y})\Big)\, d\mathring{\mu} \\ &\geq \int_{\mathcal{S}} \Big(\|\mathring{D}\boldsymbol{Y}\|^2 + 2\mathring{r}_{ij}X^iX^j + \mathring{\mathscr{R}}(\boldsymbol{X},\boldsymbol{Y})\Big)\, d\mathring{\mu}.\end{aligned} \tag{5.3.10}$$

Now,

$$2\mathring{r}_{ij}X^iX^j + \mathring{\mathscr{R}}(\boldsymbol{X},\boldsymbol{Y}) = r_{ij}X^iX^j + \tfrac{1}{2}\mathfrak{R}(\boldsymbol{Y})_{ij}Y^{ij} + 4\epsilon_{ikm}r_j{}^mX^iY^{jk}.$$

For an elliptic manifold ($\mathring{r}_{ij} = \frac{2}{3}\lambda\mathring{h}_{ij}$ with $\lambda > 0$), we obtain

$$2\mathring{r}_{ij}X^iX^j + \mathring{\mathscr{R}}(\boldsymbol{X},\boldsymbol{Y}) = \tfrac{2}{3}\lambda\|\boldsymbol{X}\|^2 + \lambda\|\boldsymbol{Y}\|^2,$$

which is positive-definite. Hence, it follows that if $\mathring{\boldsymbol{h}}$ is sufficiently positively-pinched, then the integrand in (5.3.10) is also positive-definite, which would then imply that $X_i = 0,\ Y_{ij} = 0$ —i.e. that the operator $\mathring{\mathcal{P}}^{(\alpha)}$ is injective.

The above holds in particular for $\alpha = 0, 1$, and so we obtain the following:

Proposition 16. Suppose a Riemannian metric $\mathring{\boldsymbol{h}}$ satisfies (C2)′. Then, (C4) is also satisfied provided $\mathring{\boldsymbol{h}}$ is sufficiently positively-pinched.

It is reasonable to expect that by exploiting the precise structures of $\mathring{\mathcal{P}}^{(0)}$ and $\mathring{\mathcal{P}}^{(1)}$, we may be able to relax the pinching condition and still conclude injectivity for those operators —the argument given above establishes injectivity of $\mathring{\mathcal{P}}^{(\alpha)}$ for all $\alpha \in \mathbb{R}$, which is more than we need.

5.3.4 Existence of perturbative ECE solutions for pinched background metrics

Collecting together the results of the above discussion, we have the following corollaries of Theorem 3, the proofs of which should be clear from the previous two sections.

The *negatively-pinched case*:

Corollary 2. Let $\mathring{\boldsymbol{h}}$ be a Riemannian metric on $\mathcal{S}$ with constant negative scalar curvature, $\mathring{r} = 2\lambda < 0$. There exists $\epsilon > 0$ such that, if

$$\varrho(\sigma_-, \sigma_+) \equiv \frac{1}{4\lambda^2}(\sigma_-^2 + \sigma_+^2) < \epsilon,$$

at each $p \in \mathcal{S}$, then $\mathring{\boldsymbol{h}}$ satisfies (C1)–(C3). If such a metric $\mathring{\boldsymbol{h}}$ additionally satisfies (C4), then it admits non-linear perturbative solutions of the ECEs, according to Theorem 3.

The *positively-pinched case*:

Corollary 3. Let $\mathring{\boldsymbol{h}}$ be a Riemannian metric on $\mathcal{S}$ with constant positive scalar curvature, $\mathring{r} = 2\lambda > 0$. Suppose $\mathring{\boldsymbol{h}}$ is such that either

(i) conditions (C1), (C2) and (C4) hold, **or**

(ii) conditions (C1) and (C2)$'$ hold.

Then, for case (i), there exists $\epsilon_1 > 0$ such that, if $\mathring{\boldsymbol{h}}$ additionally satisfies

$$\varrho(\sigma_-, \sigma_+) \equiv \frac{1}{4\lambda^2}(\sigma_-^2 + \sigma_+^2) < \epsilon_1, \tag{I}$$

at each $p \in \mathcal{S}$, then condition (C3) is also satisfied. For case (ii), there exists $0 < \epsilon_2 \leq \epsilon_1$ such that, if $\mathring{\boldsymbol{h}}$ additionally satisfies

$$\varrho(\sigma_-, \sigma_+) \equiv \frac{1}{4\lambda^2}(\sigma_-^2 + \sigma_+^2) < \epsilon_2, \tag{II}$$

at each $p \in \mathcal{S}$, then conditions (C3) and (C4) are also satisfied. Hence, assuming either (i) & (I), or (ii) & (II) to hold, $\mathring{\boldsymbol{h}}$ admits non-linear perturbative solutions of the ECEs, according to Theorem 3.

Comparing Corollary 2 with 3, the negatively-pinched metrics appear to be much better candidates for the Friedrich–Butscher method than the positively-pinched metrics. Analysis of condition (C4) for negatively-pinched metrics is deferred to future work.

5.4 Concluding remarks

In this chapter we have identified a set of sufficient conditions, (C1)–(C4), for a closed time symmetric initial data set to admit non-linear perturbative solutions of the the ECEs via (a slight modification of) the Friedrich–Butscher method. As in Chapter 4, the free data consists of the mean extrinsic curvature (with respect to the background metric) along with the TT components of the electric and magnetic spacetime Weyl curvatures appearing in the projected York ansatz. The main new idea here was to modify the gauge-reduction procedure so as to effect a decoupling in the linearisation of the resulting auxiliary equations. This was achieved by adding X_i-terms to the de Turck covector, thereby adapting the gauge-reduction to the perturbation of electric part of the Weyl tensor.

The conditions require that a set of five elliptic operators be injective —a second-order operator acting on each of the spaces $\mathscr{C}(\mathcal{S})$, $\Lambda^1(\mathcal{S})$, $\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})$, in addition to two first-order elliptic operators acting on $\mathcal{J}(\mathcal{S})$. The conditions were further explored in the cases of positive and negative scalar curvature. In the case of negative scalar curvature, we described how condition (C1) is trivially satisfied and condition (C2) is satisfied if the Ricci curvature tensor is, moreover, negative-definite —this can be thought of as a "negative-pinching" condition. In the case of positive scalar curvature, we found that by strengthening condition (C2) in a natural way —see (C2)$'$ in Section 5.3.3— we were able to eliminate condition (C4) in favour of a "positive-pinching" condition on the curvature of the background metric. Condition (C3), on the other hand, is guaranteed to hold if the Ricci curvature tensor is *either* negatively or positively-pinched.

So far, the only examples that we have found of manifolds satisfying all of the conditions, are the time symmetric conformally-rigid hyperbolic metrics of Chapter 4 —see Remark 35 of Section 5.3.3. It seems that the most promising candidates are the metrics of negatively-pinched curvature; as described above in Corollary 2, if the curvature is sufficiently negatively-pinched then conditions (C1)–(C3) are satisfied, leaving only condition (C4) to be analysed.

There are several other natural directions of generalisation. The most obvious is to try to extend the analysis to include non-trivial extrinsic curvature of the background. This is anticipated to be difficult due to the highly-coupled nature of the linearised equations. One could also include matter, in which case it may be easier to eliminate certain potential obstructions —i.e. to satisfy the analogues of conditions (C1)–(C4). Another method of circumventing obstructions could be to restrict to solutions of the ECEs that are invariant under the action of the group of discrete isometries of the given background solution. This method was applied to the problem of "static metric extensions" by P. Miao in [74].

It would also be interesting to see whether the analysis can also be transcribed to the asymptotically-flat setting, in which case the "$t = 0$" slice of the Kerr spacetime would serve as a natural first candidate —recall that the "$t = 0$" slice of the Schwarzschild spacetime admits tracefree Codazzi tensors (c.f. remark 13 in section 4.1.4) and therefore *does not* appear to be a good candidate. Again, this would involve the inclusion of non-trivial background extrinsic curvature.

Chapter 6

A refinement of the Friedrich–Butscher method

In the previous chapters we have followed to a large extent the method set out in [28, 29], the only deviations being the slightly modified decomposition of the magnetic and electric parts of the Weyl curvature and in the use of a generalised version of the De Turck vector in the previous chapter. The methods were developed with the aim of applying standard results of second order elliptic PDE theory. Ideally, however, the auxiliary system of equations should be constructed in such a way that any obstructions that arise are identifiable with some geometric feature of the space of solutions, rather than arising simply as an artefact of the method itself (the choice of ansatz, for instance).

In the present chapter I will propose an alternative to the Friedrich–Butscher method, which makes some progress in this direction. The new method differs from the previously described methods in two main regards. Firstly, the ECEs will be studied as a mixed-order system, making use of an inbuilt elliptic structure that has been hinted at in previous sections. Secondly, we will make use of a gauge-fixing procedure which differs from (but is based on) that of De Turck and which does not appear to be in the literature. The main advantage of the first modification is that the sufficiency argument will be greatly simplified, since we will consider the equation $J_{ijk} = 0$ directly, rather than the auxiliary equation $\mathring{\mathcal{D}}^*(\boldsymbol{J})_{ij} = 0$, while the advantage of the second modification is that it simplifies the analysis of the resulting linearised system. Given these two modifications, the cokernel of the resulting linearised secondary map admits a more systematic analysis —in particular, KID sets arise naturally as obstructions to integrability of the equations. Given the role that KID sets play in the problem of linearisation stability of the Einstein constraint equations —see Section 6.2.2— it seems that the present method may therefore be the more correct approach to studying the ECEs.

The structure of this chapter is as follows: in Section 6.1 we will discuss the generalised gauge-reduction procedure and ellipticity of the Codazzi–Mainardi equation; in Section 6.2 the simplified auxiliary system will be described, ellipticity of the linearised system addressed and the connection with KID sets discussed; in Section 6.3 we discuss applications of the IFT for the construction of solutions. The main result is Theorem 6 which is an improvement on Theorem 3 of the previous chapter in two main regards: firstly it removes condition (C2) asserting the injectivity of $\mathring{\Delta}_Y$, and secondly it removes the requirement of injectivity of the operator $\mathring{\mathcal{P}}^{(1)}$ in condition (C4). The latter is made possible through the use of a simplified (mixed-order) auxiliary ECE map the vanishing of which implies the Codazzi–Mainardi equation directly, as remarked above, rather than some auxiliary substitute (e.g. $\mathring{\mathcal{D}}^*(\boldsymbol{J})_{ij} = 0$ in previous chapters) —see the discussion in Remark 30.

6.1 Revisiting ellipticity of the ECEs

6.1.1 A modified De Turck trick

In previous chapters the Ricci operator was replaced with the reduced Ricci operator, which can be written in the form

$$\mathrm{Ric}^{\mathcal{K}}[\boldsymbol{h}]_{ij} \equiv \mathrm{Ric}[\boldsymbol{h}]_{ij} - \tfrac{1}{2}\mathcal{K}\circ Q(\boldsymbol{h})_{ij}, \tag{6.1.1}$$

with $\mathcal{K} \equiv -2\delta^*$, thereby rendering the equation $V_{ij} = 0$ (or rather, its linearisation) elliptic. More generally, one might ask for which operators $\mathcal{K} : \mathscr{S}^2(\mathcal{S}) \to \mathscr{S}^2(\mathcal{S})$ the *generalised reduced Ricci curvature* (6.1.1) has elliptic linearisation. The proof of the following proposition can be found in Appendix A.5.

Proposition 17. Let $\mathring{\mathcal{K}} : \Lambda^1(\mathcal{S}) \to \mathscr{S}^2(\mathcal{S})$ be an overdetermined-elliptic operator. Then, the operator $D\mathrm{Ric}^{\mathring{\mathcal{K}}} : \mathscr{S}^2(\mathcal{S}) \to \mathscr{S}^2(\mathcal{S})$ defined by

$$D\mathrm{Ric}^{\mathring{\mathcal{K}}} \equiv D\mathrm{Ric} - \tfrac{1}{2}\mathring{\mathcal{K}}\circ \mathring{B},$$

is elliptic if and only if $\mathring{B}\circ\mathring{\mathcal{K}}$ is elliptic.

Recall that $B : \mathscr{S}^2(\mathcal{S}) \to \Lambda^1(\mathcal{S})$ denotes the Bianchi operator, acting as

$$B(\boldsymbol{\gamma})_i \equiv \delta(\boldsymbol{\gamma})_i - \tfrac{1}{2}d(\mathrm{tr}_{\boldsymbol{h}}\boldsymbol{\gamma})_i.$$

Remark 38. Note that a gauge-reduced equation of the form

$$\mathrm{Ric}^{\mathcal{K}}[\boldsymbol{h}]_{ij} = F_{ij},$$

results in an integrability condition of the form

$$B\circ\mathcal{K}(Q)_i = -2B(\boldsymbol{F})_i,$$

for the quantity $Q(\boldsymbol{h})_i$, as a consequence of the contracted Bianchi identity. Proposition 17 can therefore be thought of as establishing equivalence of ellipticity of the reduced operator and ellipticity of the associated integrability condition. Note that the de Turck trick is recovered by setting $\mathcal{K} = -2\delta^*$.

As a result, if we choose an operator $\mathcal{K} : \Lambda^1(\mathcal{S}) \to \mathscr{S}^2(\mathcal{S})$ for which $B\circ\mathcal{K}$ is elliptic, then

$$\mathrm{Ric}^{\mathcal{K}}[\boldsymbol{h}]_{ij} = \mathrm{Ric}[\boldsymbol{h}]_{ij} - \tfrac{1}{2}\mathcal{K}_{\boldsymbol{h}}(\boldsymbol{Q}(\boldsymbol{h}))_{ij}$$

has elliptic (though not necessarily self-adjoint) linearisation $D\mathrm{Ric}^{\mathring{\mathcal{K}}}$. In particular, we shall see that the choice $\mathcal{K} = L$ is particularly well-suited to the Friedrich–Butscher method, since it effects a semi-decoupling of the linearised equations —recall that L (the conformal Killing operator) is overdetermined elliptic and that

$$B\circ L \equiv \delta\circ L,$$

which is is elliptic as discussed in previous chapters, and therefore $\mathcal{K} = L$ indeed satisfies the conditions of the proposition. As we shall see in Section 6.2.2, the use of $\mathcal{K} = L$ in the gauge-reduction has the added benefit that leaves the linearised *scalar* curvature unaffected.[1] Let us verify

[1]This holds more generally for any $\mathcal{K}$ with image in $\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$.

ellipticity of the operator

$$\mathrm{Ric}^L[\boldsymbol{h}]_{ij} \equiv \mathrm{Ric}[\boldsymbol{h}]_{ij} - \tfrac{1}{2} L \circ Q(\boldsymbol{h})_{ij}$$

directly. Written in terms of the trace and tracefree parts of γ_{ij} with respect to $\mathring{\boldsymbol{h}}$, denoted $\gamma, \bar{\gamma}_{ij}$, we have

$$\begin{aligned} D\mathrm{Ric}^L(\boldsymbol{\gamma})_{ij} &= D\mathrm{Ric}(\boldsymbol{\gamma})_{ij} - \tfrac{1}{2}\mathring{L} \circ \mathring{B}(\boldsymbol{\gamma})_{ij} \\ &= \tfrac{1}{2}\mathring{\Delta}_L \bar{\gamma}_{ij} + \tfrac{1}{9}(3\mathring{\delta} \circ \mathring{\delta}(\bar{\boldsymbol{\gamma}}) - 2\mathring{\Delta}\gamma)\mathring{h}_{ij}. \end{aligned}$$

Taking the trace and tracefree parts with respect to $\mathring{\boldsymbol{h}}$, $D\mathrm{Ric}^L(\boldsymbol{\gamma})_{ij} = 0$ is equivalent to

$$\begin{aligned} &\mathring{\Delta}_L \bar{\gamma}_{ij} = 0, \\ &\mathring{\Delta}\gamma - \tfrac{3}{2}\mathring{\delta} \circ \mathring{\delta}(\bar{\gamma}) = 0, \end{aligned}$$

with corresponding symbol maps

$$\begin{aligned} &|\boldsymbol{\xi}|^2 \bar{\gamma}_{ij} = 0, \\ &|\boldsymbol{\xi}|^2 \gamma - \tfrac{3}{2}\xi^i \xi^j \bar{\gamma}_{ij} = 0. \end{aligned}$$

The first equation immediately implies that $\bar{\gamma}_{ij} = 0$ —i.e. $\mathring{\Delta}_L$ is elliptic. Substituting into the second equation, we find that $\gamma = 0$. Hence, the operator $D\mathrm{Ric}^L(\cdot)$ is indeed elliptic.

Moreover, we will see that the choice $\mathcal{K} = L$ leads to a particularly simple sufficiency argument. In particular, since $B \circ L \equiv \delta \circ L \equiv -\frac{1}{2}L^* \circ L$ one only requires non-existence of conformal Killing vectors to conclude vanishing of the De Turck vector; the details are given in Proposition 19. In contrast, previous approaches required the added restriction that $\mathring{\boldsymbol{h}}$ admit no infinitesimal harmonic deformations (i.e. non-trivial elements of ker $\mathring{\Delta}_Y$).

As before, we have the freedom to add an arbitrary vector field to the definition of the de Turck vector. Again it is convenient to use the covector field $Q^{\boldsymbol{X}}(\boldsymbol{h}) \equiv Q(\boldsymbol{h}) + 2(\mathring{\boldsymbol{X}} - \boldsymbol{X})$. Accordingly, we define

$$\mathrm{Ric}^{L,\boldsymbol{X}}[\boldsymbol{h}]_{ij} = \mathrm{Ric}[\boldsymbol{h}]_{ij} - \tfrac{1}{2}L(Q^{\boldsymbol{X}}(\boldsymbol{h}) + 2(\mathring{\boldsymbol{X}} - \boldsymbol{X}))_{ij},$$

the full linearisation of which will be given in Section 6.2.1.

6.1.2 Revisiting ellipticity of the Codazzi–Mainardi equation

In this section, we will exhibit an elliptic structure within the Codazzi–Mainardi equation. More precisely it is shown that, when the fields K_{ij} and $\bar{S}_{ij}$ are decomposed in the appropriate way, the Codazzi–Mainardi equation can be thought of as a first-order elliptic equation for a subset of the unknown fields, with principal part given by the operator $\mathcal{P}^{(0)}$ (or, equivalently, $\mathcal{K}^{(0)}$) from Section 5.1.2. The discussion of this section should be compared with that of Section 5.1.3 —see, in particular, Lemma 11.

Recall the Codazzi–Mainardi equation:

$$J_{ijk} \equiv D_i K_{jk} - D_j K_{ik} - \epsilon_{ij}{}^l \bar{S}_{kl} = 0.$$

Let us decompose

$$K_{ij} = F_{ij} + \tfrac{1}{3} f h_{ij},$$

in addition to performing a York split of the magnetic tensor:

$$\bar{S}_{ij} = L(\bar{\boldsymbol{A}})_{ij} + \bar{B}_{ij},$$

where L is again the conformal Killing operator, acting here on $A_i \in \Lambda^1(\mathcal{S})$, and $\bar{B}_{ij} \in \mathscr{S}_{TT}(\mathcal{S}; \boldsymbol{h})$. Substituting the above decompositions, (3.3.9a) may be read as an (inhomogeneous) equation for F_{ij} and $\bar{A}_i$ as follows

$$D_i F_{jk} - D_j F_{ik} - \epsilon^l{}_{ij} L(\bar{\boldsymbol{A}})_{kl} = -\tfrac{1}{3}(df)_i h_{jk} + \tfrac{1}{3}(df)_j h_{ik} + \epsilon^l{}_{ij} \bar{B}_{kl}.$$

Note that the left-hand-side is an element of $\mathcal{J}(\mathcal{S})$. Performing the *Jacobi decomposition* with respect to $\boldsymbol{h}$ one obtains

$$\mathcal{P}^{(0)} \begin{pmatrix} F_{ij} \\ -2\bar{A}_i \end{pmatrix} = \begin{pmatrix} 2\bar{B}_{ij} \\ \frac{2}{3}(df)_i \end{pmatrix}$$

which is first-order elliptic as an equation for $F_{ij},\ \bar{A}_i$. Alternatively, if one defines

$$\mu_{ijk} = \tfrac{1}{2}(\epsilon_{ij}{}^l F_{kl} - 2\bar{A}_i h_{jk} + 2\bar{A}_j h_{ik}) \in \mathcal{J}(\mathcal{S}),$$

the above can be expressed equivalently as follows

$$\mathcal{K}^{(0)}(\boldsymbol{\mu})_{ijk} = -\tfrac{1}{2}(df)_i h_{jk} + \tfrac{1}{3}(df)_j h_{ik} + \epsilon^l{}_{ij} \bar{B}_{kl}, \tag{6.1.2}$$

where $\mathcal{K}^{(0)} : \mathcal{J}(\mathcal{S}) \to \mathcal{J}(\mathcal{S})$ is the first-order elliptic operator given in Section 5.1.2. Hence, the Codazzi–Mainardi equation is already (first-order) elliptic when thought of as an equation for the unknown fields F_{ij} and $\bar{A}_i$, with the fields $f,\ \bar{B}_{ij}$ prescribed.

Since our method of solving the ECEs will again be perturbative, we will use the same ansatz for $K_{ij},\ \bar{S}_{ij}$ as in Chapters 4 and 5, rather than the above decompositions. It is clear that when equation (6.1.2) is linearised, the principal part will consist of the operator $\mathring{\mathcal{K}}^{(0)}$ and so the equation will again be elliptic.

Remark 39. Note that in the full CCEs, the Codazzi–Mainardi equation is replaced with the following equation ($Y_{ijk} = 0$):

$$D_i K_{jk} - D_j K_{ik} - \Omega \epsilon_{ij}{}^l d^*_{kl} = h_{ik} L_j - h_{jk} L_i.$$

If we were to follow the same approach as outlined above, the principal part (which would now contain both $\mathcal{D}(\boldsymbol{K})-$terms and $\Omega L(\bar{\boldsymbol{X}})-$terms) would degenerate at the boundary $\partial\mathcal{S}$, where $\Omega = 0$. It seems therefore that one must follow an approach more in keeping with the Friedrich–Butscher method, as described in the previous two chapters —one applies $\mathring{\mathcal{D}}^*$ to the equation to obtain a second-order equation for the principal variables $\chi_{ij} \equiv K_{ij} - \frac{1}{2}(\mathrm{tr}_{\mathring{\boldsymbol{h}}} \boldsymbol{K})\mathring{h}_{ij}$. This will be explored in more detail in Chapter 7.

6.2 The simplified auxiliary system

Following from the discussion in the previous sections, we now define the auxiliary map as follows [2]

$$\widetilde{\Psi}(\boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}, \boldsymbol{h}; \phi, \bar{\boldsymbol{T}}, \boldsymbol{T}) \equiv$$
$$\begin{pmatrix} J_{ijk} \\ \Lambda_i \\ V_{ij} - \frac{1}{2}L(Q^{\boldsymbol{X}}(\boldsymbol{h}))_{ij} \end{pmatrix} = \begin{pmatrix} \big(\mathcal{D}(\boldsymbol{K}) - \star\bar{\boldsymbol{S}}\big)_{ijk} \\ \delta(\boldsymbol{S})_i + \epsilon^{jk}{}_i K_j{}^l \bar{S}_{kl} \\ \mathrm{Ric}^{L,\boldsymbol{X}}[\boldsymbol{h}]_{ij} - \frac{2}{3}\lambda h_{ij} - S_{ij} + KK_{ij} - K_i{}^k K_{jk} \end{pmatrix},$$

with the same ansatz as in Chapters 4 and 5, namely

$$K_{ij} = \chi_{ij} + \tfrac{1}{3}\phi\mathring{h}_{ij} \qquad S_{ij} = \Pi_{\boldsymbol{h}}(\mathring{L}(\boldsymbol{X}) + \boldsymbol{T})_{ij}, \qquad \bar{S}_{ij} = \Pi_{\boldsymbol{h}}(\mathring{L}(\bar{\boldsymbol{X}}) + \bar{\boldsymbol{T}})_{ij},$$

with $\chi_{ij} \in \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})$, $\bar{T}_{ij}, T_{ij} \in \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})$. As before,

$$u \equiv (\phi, \bar{\boldsymbol{T}}, \boldsymbol{T}), \qquad v \equiv (\boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}, \boldsymbol{h})$$

will be considered the free data and the determined fields, respectively, and we will denote the linearisations in the direction of the free and determined fields by $D_u\widetilde{\Psi}$, $D_v\widetilde{\Psi}$, respectively.

It should be noted that, since we are now working directly with the equation $J_{ijk} = 0$ rather than $\mathring{\mathcal{D}}^*(\boldsymbol{J})_{ij} = 0$, we no longer require the equation $\bar{\Lambda}_i = 0$ —in fact, it will automatically be satisfied by virtue of its role as an integrability condition.

6.2.1 Ellipticity of the linearised auxiliary map

Recall that in previous chapters ellipticity of the relevant linearised auxiliary equations was central to the application of the IFT. Here we show that the operator $D_v\widetilde{\Psi}$ is again elliptic, though in a weaker sense than considered previously.

We first compute the linearisation of the reduced Ricci operator $\mathrm{Ric}^{L,\boldsymbol{X}}$:

$$\begin{aligned}
D\mathrm{Ric}^{L,\boldsymbol{X}}(\boldsymbol{\gamma}, \boldsymbol{\xi})_{ij} &\equiv \frac{d}{d\tau}\mathrm{Ric}^{L,\boldsymbol{X}+\tau\boldsymbol{\xi}}[\boldsymbol{h} + \tau\boldsymbol{\gamma}]_{ij}\Big|_{\tau=0} \\
&= \frac{d}{d\tau}\mathrm{Ric}^{L,\mathring{\boldsymbol{X}}}[\boldsymbol{h} + \tau\boldsymbol{\gamma}]_{ij}\Big|_{\tau=0} + \frac{d}{d\tau}\mathrm{Ric}^{L,\boldsymbol{X}+\tau\boldsymbol{\xi}}[\mathring{\boldsymbol{h}}]_{ij}\Big|_{\tau=0} \\
&= D\mathrm{Ric}^L(\boldsymbol{\gamma})_{ij} + \frac{d}{d\tau}L_{\mathring{\boldsymbol{h}}+\tau\gamma}(\mathring{\boldsymbol{X}} + \tau\boldsymbol{\xi} - \mathring{\boldsymbol{X}}_{ij})\Big|_{\tau=0} \\
&= D\mathrm{Ric}^L(\boldsymbol{\gamma})_{ij} + \frac{d}{d\tau}L_{\mathring{\boldsymbol{h}}+\tau\gamma}(\tau\boldsymbol{\xi})_{ij}\Big|_{\tau=0} \\
&= D\mathrm{Ric}^L(\boldsymbol{\gamma})_{ij} + \mathring{L}(\boldsymbol{\xi})_{ij} \\
&= \tfrac{1}{2}\mathring{\Delta}_L\bar{\gamma}_{ij} + \tfrac{1}{9}(3\mathring{\delta}\circ\mathring{\delta}(\bar{\boldsymbol{\gamma}}) - 2\mathring{\Delta}\gamma)\mathring{h}_{ij} + \mathring{L}(\boldsymbol{\xi})_{ij},
\end{aligned}$$

where, in particular, we are using the fact that

$$\mathrm{Ric}^{L,\mathring{\boldsymbol{X}}}[\boldsymbol{h}]_{ij} \equiv \mathrm{Ric}^L[\boldsymbol{h}]_{ij}.$$

[2] Note that although we use the same symbol, $\widetilde{\Psi}$, the auxiliary map differs from those considered in previous chapters.

The linearisation of the auxiliary map in the direction of the determined fields is then given by

$$D_v\widetilde{\Psi}(\boldsymbol{\chi},\bar{\boldsymbol{\xi}},\boldsymbol{\xi},\gamma) \equiv \frac{d}{d\tau}\widetilde{\Psi}(\mathring{\boldsymbol{\chi}}+\tau\boldsymbol{\sigma},\mathring{\bar{\boldsymbol{X}}}+\tau\bar{\boldsymbol{\xi}},\mathring{\boldsymbol{X}}+\tau\boldsymbol{\xi},\mathring{\boldsymbol{h}}+\tau\gamma;\mathring{\phi},\mathring{\bar{\boldsymbol{T}}},\mathring{\boldsymbol{T}})\bigg|_{\tau=0}$$

$$= \begin{pmatrix} \mathring{\mathcal{D}}(\boldsymbol{\chi})_{ijk} - \mathring{\epsilon}_{ij}{}^{l}\mathring{L}(\bar{\boldsymbol{\xi}})_{kl} + \mathscr{F}(\boldsymbol{\gamma})_{ijk} \\ \mathring{\delta}\circ\mathring{L}(\boldsymbol{\xi})_i - \mathring{\epsilon}_{ikl}\mathring{\bar{S}}^{jl}\chi_j{}^{k} + \mathring{\epsilon}_{ijl}\mathring{K}_k{}^{l}\mathring{L}(\bar{\boldsymbol{\xi}})^{jk} + \mathscr{F}_3(\boldsymbol{\gamma})_i \\ \frac{1}{2}\mathring{\Delta}_L\gamma_{ij} + \frac{1}{3}\mathring{\delta}\circ\mathring{\delta}(\boldsymbol{\gamma})\mathring{h}_{ij} - \frac{1}{6}\mathring{h}_{ij}\mathring{\Delta}\gamma - \frac{2}{3}\lambda\gamma_{ij} + \mathscr{F}_4(\boldsymbol{\chi})_{ij} + \mathscr{F}_5(\boldsymbol{\gamma})_{ij} \end{pmatrix} \tag{6.2.1}$$

where

$$\begin{aligned}
\mathscr{F}(\boldsymbol{\gamma})_{ijk} &= -\mathring{K}_j{}^{l}C(\boldsymbol{\gamma})_{lik} + \mathring{K}_i{}^{l}C(\boldsymbol{\gamma})_{ljk} - \tfrac{1}{3}\mathring{\epsilon}_{ijk}\check{S}^{lm}\gamma_{lm} + \mathring{\epsilon}_{ijm}\mathring{\bar{S}}_k{}^{l}\gamma_l{}^{m} - \tfrac{1}{2}\mathring{\epsilon}_{ijl}\mathring{\bar{S}}_k{}^{l}\gamma, \\
\mathscr{F}_3(\boldsymbol{\gamma})_i &\equiv -\mathring{S}_i{}^{j}\mathring{B}(\boldsymbol{\gamma})_j + \mathring{\epsilon}_{ikm}\mathring{K}^{jk}\mathring{\bar{S}}^{lm}\gamma_{jl} - \mathring{\epsilon}_{ilm}\mathring{K}^{jk}\mathring{\bar{S}}_j{}^{l}\gamma_k{}^{m} + \mathring{\epsilon}_{ikm}\mathring{K}^{jk}\mathring{\bar{S}}_j{}^{l}\gamma_l{}^{m} \\
&\qquad - \tfrac{1}{2}\mathring{\epsilon}_{ikl}\mathring{K}^{jk}\mathring{\bar{S}}_j{}^{l}\gamma + \tfrac{1}{3}\gamma^{jk}\mathring{D}_i\mathring{S}_{jk} - \tfrac{1}{6}\mathring{S}^{jk}\mathring{D}_i\gamma_{jk} - \gamma^{jk}\mathring{D}_k\mathring{S}_{ij}, \\
\mathscr{F}_4(\boldsymbol{\chi})_{ij} &= \mathring{K}\chi_{ij} - 2\mathring{K}_{(i}{}^{k}\chi_{j)k}, \\
\mathscr{F}_5(\boldsymbol{\gamma})_{ij} &= \mathring{K}_i{}^{k}\mathring{K}_j{}^{l}\gamma_{kl} - \mathring{K}_{ij}\mathring{K}^{kl}\gamma_{kl} - \tfrac{1}{3}\mathring{S}^{kl}\gamma_{kl}\mathring{h}_{ij}.
\end{aligned}$$

In previous sections we have discussed ellipticity of each of the operators $\mathring{\mathcal{P}}^{(\alpha)}$, $\mathring{\delta}\circ\mathring{L}$, $D\mathrm{Ric}^{L,\boldsymbol{X}}$. While the first is first-order elliptic, the second and third are second-order elliptic. We would like to know then whether the operator $D_v\widetilde{\Psi}$, as a whole, can be considered *mixed-order* elliptic, in some sense. One such notation is that of *Douglis–Nirenberg* ellipticity, first introduced in [75, 76]:

Definition 12. A linear differential operator, $P: E \to E$, with E a bundle of dimension N over a manifold $\mathcal{S}$ of dimension $n \geq 3$, can be written in local coordinates in the form

$$P(x,\partial)_{\mu\nu}u^{\nu}(x) = f_{\mu}(x), \qquad \mu,\nu = 1,\cdots N,$$

with $[P_{\mu\nu}]$ an $N\times N$ matrix whose components are polynomials in $\partial_1,\cdots,\partial_n$, with coefficients depending on x. Let $s_1\cdots s_N$ and $t_1,\cdots t_N$ be integers —the *Douglis–Nirenberg* (D–N) *weights*— such that for each μ,ν,

$$\deg(P_{\mu\nu}) \leq s_\mu + t_\nu,$$

where "deg" is the degree as a polynomial of the partial derivatives ∂. For given D–N weights, the principal part $p(x,\partial)$ consists of those terms of $P_{\mu\nu}$ which are precisely of order $s_\mu + t_\nu$. The operator $p(x,\partial)_{\mu\nu}$ is *elliptic* if its principal symbol map $p(x,\boldsymbol{\xi})_{\mu\nu}$ is injective for each $\xi_i \in \Lambda^1(\mathcal{S})$. Finally, P is said to be *Douglis–Nirenberg elliptic* if there exists a choice of D–N weights for which the corresponding principal part is elliptic.

Remark 40. By adding the same constant to each of t_ν and subtracting it from each s_μ, it is clearly possible to arrange that max $s_\mu = 0$. If P is Douglis–Nirenberg elliptic then so is its adjoint. To see this, note that, writing

$$P(x,D)_{\mu\nu} = \sum_{|\rho|=0}^{s_\mu+t_\nu} A_{\mu\nu,\rho}D^{\rho},$$

where ρ is a multi-index, the formal L^2- adjoint takes the form

$$P(x,D)^{*}_{\mu\nu} = \sum_{|\rho|=s_\mu+t_\nu} (-1)^{s_\mu+t_\nu}A_{\mu\nu,\rho}D^{\rho} + \sum_{|\rho|=0}^{s_\mu+t_\nu-1} B_{\mu\nu,\rho}D^{\rho}$$

for some $B_{\mu\nu,\rho}$. Then taking the same weights s_μ, t_ν the principal symbol is

$$(-1)^{s_\mu+t_\nu}\xi^\rho A_{\mu\nu,\rho}$$

which has

$$\det((-1)^{s_\mu+t_\nu}\xi^\rho A_{\mu\nu,\rho}) = (-1)^{s+t}\det(\xi^\rho A_{\mu\nu,\rho}) \neq 0,$$

with $s \equiv s_1 + \cdots s_N$ and $t \equiv t_1 + \cdots t_N$.

Example 1. The prototypical example (see e.g. [44, 76]) of a Douglis–Nirenberg operator is the scalar Laplacian on $\mathbb{R}^2$, $\Delta_{\boldsymbol{\delta}} = \partial_1^2 + \partial_2^2$, thought of as the following first-order operator

$$\begin{pmatrix} 0 & \partial_1 & \partial_2 \\ \partial_1 & -1 & 0 \\ 0 & \partial_2 & -1 \end{pmatrix}\begin{pmatrix} u \\ u_1 \\ u_2 \end{pmatrix}.$$

Choosing weights $t_1 = 2,\ t_2 = t_3 = 1$ and $s_1 = 0,\ s_2 = s_3 = -1$, the principal symbol is

$$\begin{pmatrix} 0 & \xi_1 & \xi_2 \\ \xi_1 & -1 & 0 \\ 0 & \xi_2 & -1 \end{pmatrix}$$

which has determinant $\xi_1^2 + \xi_2^2$, and is therefore injective.

Proposition 18. The operator $D_v\widetilde{\Psi}$ is Douglis–Nirenberg elliptic.

Proof. We would like to choose Douglis–Nirenberg weights for which the resulting principal part of $D_v\widetilde{\Psi}$ is of the form

$$\begin{pmatrix} 2\mathring{\mathcal{R}} & -2\mathring{L} & 0 & 0 \\ -2\mathring{\delta} & 0 & 0 & 0 \\ 0 & 0 & \mathring{\delta}\circ\mathring{L} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\mathring{\Delta} + \frac{1}{3}\mathring{\boldsymbol{h}}\mathring{\delta}\circ\mathring{B} \end{pmatrix}. \tag{6.2.2}$$

First note that $D_v\widetilde{\Psi}$ can be written as a matrix as follows

	t_1	t_2	t_3	t_4
s_1	$2\mathring{\mathcal{R}}(\cdot)$	$-2\mathring{L}(\cdot)$	0	$\mathscr{F}_1(\cdot)$
s_2	$\mathring{\delta}(\cdot)$	0	0	$\mathscr{F}_2(\cdot)$
s_3	$-\mathring{\bar{\boldsymbol{S}}}\times(\cdot)$	$\mathring{\boldsymbol{K}}\times\mathring{L}(\cdot)$	$\mathring{\delta}\circ\mathring{L}(\cdot)$	$\mathscr{F}_3(\cdot)$
s_4	$\mathscr{F}_4(\cdot)$	0	0	$\left(-\frac{1}{2}\mathring{\Delta} + \frac{1}{3}\mathring{\boldsymbol{h}}\mathring{\delta}\circ\mathring{B} + \mathscr{F}_5\right)(\cdot)$

where

$$\mathscr{F}_1(\boldsymbol{\gamma}) = \tfrac{1}{2}\epsilon_{kl(i}\mathscr{F}(\boldsymbol{\gamma})^{kl}{}_{j)}, \qquad \mathscr{F}_2(\boldsymbol{\gamma})_j = \mathscr{F}(\boldsymbol{\gamma})_{ij}{}^{i},$$

and "$\mathring{\boldsymbol{K}}\times\mathring{L}(\cdot)$" is being used as a shorthand for the operator

$$\mathring{\boldsymbol{K}}\times\mathring{L}(\bar{\boldsymbol{\xi}})_i = \mathring{\epsilon}_{ikl}\mathring{K}_j{}^l\mathring{L}(\bar{\boldsymbol{\xi}})^{kj}.$$

We see then that (6.2.2) is the principal part of $D_v\widetilde{\Psi}$ for any choice of weights satisfying the relations

$$\begin{array}{llll} s_1+t_1=1, & s_1+t_2=1, & s_1+t_3\geq 0, & s_1+t_4>1, \\ s_2+t_1=1, & s_2+t_2\geq 0, & s_2+t_3\geq 0, & s_2+t_4>1, \\ s_3+t_1>0, & s_3+t_2>1, & s_3+t_3=2, & s_3+t_4>1, \\ s_4+t_1>0, & s_4+t_2\geq 0, & s_4+t_3\geq 0, & s_4+t_4=2. \end{array}$$

In particular,

$$s_1=s_2=s_4=-1, \quad s_3=0, \qquad t_1=t_2=t_3=2, \quad t_4=3$$

is a suitable choice of weights. Clearly (6.2.2) has injective symbol by ellipticity of each of the constituent operators, and hence $D_v\widetilde{\Psi}$ is Douglis–Nirenberg elliptic. □

Basic Fredholm Theory for Douglis–Nirenberg systems

In order to prove the existence of solutions to the auxiliary equations, $\widetilde{\Psi}=0$, we first need to show that the operator $D_v\widetilde{\Psi}$ is Fredholm. The first step is to prove an elliptic estimate for Douglis–Nirenberg operators.

Consider a system of equations of the form

$$P(x,D)_{\mu\nu}u^\nu = F(x)_\mu \tag{6.2.3}$$

where we are using the summation convention over $\nu=1,\cdots,N$ and write

$$P(x,D)_{\mu\nu} = \sum_{|\rho|=0}^{s_\mu+t_\nu} A_{\mu\nu,\rho}D^\rho$$

with ρ a multi-index. In the following, we will use $\|\cdot\|_k$ and $\|\cdot\|_{\tilde{H}^k}$ to denote the supremum $k-$norm and Sobolev $k-$norm, respectively, on $\mathbb{R}^m$ with respect to the flat metric and connection.

In Theorem 10.3 of [76], the authors prove the following "interior elliptic estimate":

Theorem 4. Let $P(x,\partial)$ be a Douglis–Nirenberg elliptic operator with weights s_μ, t_ν, $\max s_\mu=0$, with $\|A_{\mu\nu,\rho}\|_{l-s_\mu}<\infty$, for $\nu=1,\cdots,N$ and some $l\geq 0$. Let $u(x)^\nu$ satisfy (6.2.3), where we assume that $\|F(x)_\mu\|_{l-s_\mu}<\infty$. Then, there exist positive constants r_l, K_l such that, if $u(x)^\nu$ is compactly supported on a sphere of radius r_l and $\|u^\nu\|_{\tilde{H}^{t_\nu}}<\infty$ for $\nu=1,\cdots,N$, then $\|u^\nu\|_{\tilde{H}^{l+t_\nu}}<\infty$ and

$$\|u^\nu\|_{\tilde{H}^{l+t_\nu}} \leq K_l\left(\sum_{\mu=1}^N \|F_\mu\|_{\tilde{H}^{l-s_\mu}} + \sum_{\mu=1}^N \|u^\mu\|_{\tilde{L}^2}\right). \tag{6.2.4}$$

In order to obtain the analogous result for an Douglis–Nirenberg operator defined over a closed Riemannian manifold, we "patch together" the above interior estimate. For $l\geq 0$, define the Banach spaces

$$\mathcal{B}_1^l \equiv H^{l+t_1}\times\cdots\times H^{l+t_N}, \qquad \mathcal{B}_2^l \equiv H^{l-s_1}\times\cdots\times H^{l-s_2}.$$

Theorem 5. Let $(\mathcal{S},\boldsymbol{h})$ be a smooth closed Riemannian manifold and $P(x,D)_{\mu\nu}$ a Douglis–Nirenberg operator with weights s_μ and t_ν. Then, given a non-negative integer l, there exists a positive constant

C_l such that, if $u^\sigma \in H^{t_\sigma}$ is a solution to (6.2.3) with $F_\mu \in C^{l-s_\mu}$, then $u^\sigma \in H^{l+t_\sigma}$ and

$$\|u^\sigma\|_{H^{l+t_\sigma}} \leq C_l \left(\sum_{\mu=1}^{N} \|F_\mu\|_{H^{l-s_\mu}} + \sum_{\nu=1}^{N} \|u^\nu\|_{H^{1+t_\nu-1}} \right). \tag{6.2.5}$$

It follows that the operator $P : B_1^l \to B_2^l$ is Fredholm. More specifically,

(i) ker $P|_{\mathcal{B}_1^l}$ is closed and finite dimensional, and Im $P|_{\mathcal{B}_1^l}$ is closed in $\mathcal{B}_2^l$;

(ii) coker P is closed in B_2^l and finite-dimensional, and is isomorphic to ker $P^*|_{\mathcal{B}_2^l}$.

For a proof sketch, see Appendix A.2.

Remark 41. We again have a statement of elliptic regularity: if, given $l \geq 1$, $u^\nu \in H^{l+t_\nu-1}$ for each $\nu = 1, \cdots, N$ and the tuple $(u^1, \cdots, u^N)$ is in the kernel of P, then the estimate (6.2.5) implies that $u^\nu \in H^{l+t_\nu}$ for each $\nu = 1, \cdots, N$. Iterating, it follows that $u^\nu \in H^k$ for any given k and the Sobolev embedding theorem implies that the functions u^ν are smooth.

It is not clear whether an analogous result exists for weighted Sobolev spaces, H^s_δ —see Chapter 8 for a definition of H^s_δ. This would be required if one wished to construct asymptotically-Euclidean initial data sets by the method proposed here.

6.2.2 Obstructions to the existence of solutions, revisited

In this section, we return to the issue of identifying the obstructions to the construction of solutions of $\widetilde{\Psi} = 0$. Recall that in Chapter 4 it was shown that, for an umbilical background initial data set, the existence of conformal Killing vectors or tracefree Codazzi tensors implies non-invertibility of the auxiliary extended constraint map. Therefore, if such non-trivial tensors exist, then they obstruct the application of the IFT. In Theorem 3 of Chapter 5 it was shown in particular that, for a time symmetric background initial data set, the kernel and cokernel of the linearised auxiliary map trivialise if the operators $\mathring{\Delta} + \lambda,\ \mathring{\Delta}_Y,\ \mathring{P}_L,\ \mathring{\mathcal{P}}^{(0)},\ \mathring{\mathcal{P}}^{(1)}$ are injective on their respective spaces of smooth sections.

We will see in this section that, under the proposed refinement of the Friedrich–Butscher method, *Killing Initial Data* (*KID*) sets arise naturally as obstructions to integrability of the auxiliary system, $\widetilde{\Psi} = 0$. We first present here some relevant background on KID sets. Recall that KID sets are "lapse-shift" pairs $(\mathcal{N}, Y_i) \in \mathscr{C}(\mathcal{S}) \times \Lambda^1(\mathcal{S})$ satisfying the KID equations, which in the vacuum case (with cosmological constant) read:

$$\mathcal{N} K_{ij} + D_{(i} Y_{j)} = 0, \tag{6.2.6a}$$

$$\mathcal{L}_{\boldsymbol{Y}} K_{ij} + D_i D_j \mathcal{N} - \mathcal{N}(r_{ij} + K K_{ij} - 2K_{ik}K^k{}_j - \lambda h_{ij}) = 0. \tag{6.2.6b}$$

The connection with the Einstein constraint equations is through the notion of *linearisation stability*. See [77], or [78] for a more succinct overview.

Definition 13. Given a map of Banach spaces, $\Phi : \mathcal{Y} \to \mathcal{Z}$, the system of equations $\Phi(x) = 0$ is said to be *linearisation stable* at $x_0 \in$ ker Φ if every element of the kernel of the linearised map $D\Phi(x_0) : \mathcal{Y} \to \mathcal{Z}$ is tangent in the Banach space $\mathcal{X}$ to a curve of solutions to $\Phi = 0$ —that is to say, if for each $X \in$ ker $D\Phi(x_0)$ there exists $\varepsilon > 0$ and a differentiable curve $x(t),\ t \in [0, \varepsilon)$ with $x(0) = x_0$, $\Phi(x(t)) = 0\ \forall\ t \in [0, \varepsilon)$, and such that $x'(0) = X$.

From now on Φ will (as before) refer specifically to the Einstein constraint map. As remarked in the Introduction, the KID equations are equivalent to $D\Phi^* = 0$, with the latter given explicitly by

$$D\Phi^* \begin{pmatrix} X \\ X_i \end{pmatrix} = \begin{pmatrix} D_i D_j X - \Delta_h X h_{ij} + X(2K^k{}_i K_{jk} - 2KK_{ij} - r_{ij}) \\ -\frac{1}{2}\mathcal{L}_X K_{ij} + \frac{1}{2}K_{ij}D_k X^k + \frac{1}{2}K_{kl}D^k X^l h_{ij} + \frac{1}{2}X^k D_k K h_{ij} \\ D_{(i}X_{j)} - D^k X_k h_{ij} + 2X(Kh_{ij} - K_{ij}) \end{pmatrix}. \qquad (6.2.7)$$

As a consequence of Lemma 3 (see Section 2.3), and using the fact that $D\Phi$ is underdetermined elliptic, we have the following (see [14]):

Lemma. If $\mathcal{S}$ is a compact manifold, then Φ is a mapping from $(\boldsymbol{h}, \boldsymbol{K}) \in M^p_{s+2} \times W^p_{s+1}$ into W^p_s, if $p > \frac{n}{2}$, $s \geq 0$, and it holds that:

$$W^p_s = \text{Im } D\Phi \oplus \ker\,(D\Phi)^*,$$

and

$$W^p_s = \ker\, D\Phi \oplus \text{Im } (D\Phi)^*,$$

where the splitting is L^2-orthogonal.

Here, M^p_{s+2} is the subspace of $W^p_{s+2}(\mathscr{S}^2(\mathcal{S}))$ consisting of $(0,2)$ tensors which are positive-definite at each $p \in \mathcal{S}$. The *Fischer–Marsden criterion* (see [77]) —namely that Φ is linearisation stable at $(\boldsymbol{h}, \boldsymbol{K})$ if and only if $(D\Phi)^*$ is injective— then follows from the above lemma and the Implicit Function Theorem.

It was later shown by Moncrief that the equations $(D\Phi)^* = 0$ (which may be simplified to the above KID equations) are exactly the conditions for $\mathcal{N},\ Y^i$ to be initial data for the lapse and shift of a spacetime Killing vector, constructed as the solution to a certain hyperbolic PDE —see [8]. The following theorem follows:

Theorem. (Moncrief, [8]) A closed vacuum initial data set $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$ with $(\boldsymbol{h}, \boldsymbol{K}) \in M_2{}^p \times W^p_1$, $p > \frac{n}{2}$, is linearisation stable if its resulting (vacuum) spacetime development admits no Killing vector field.

Let us now return to the auxiliary ECE map $\widetilde{\Psi}$. After a lengthy computation the L^2-adjoint of $D_v\widetilde{\Psi}$, acting on an arbitrary section

$$(\rho_{ijk}, \varsigma_i, \eta_{ij}) \in \mathcal{J}(\mathcal{S}) \times \Lambda^1(\mathcal{S}) \times \mathscr{S}^2(\mathcal{S}),$$

is found to be

$$(D_v\widetilde{\Psi})^*(\boldsymbol{\rho}, \boldsymbol{\varsigma}, \boldsymbol{\eta}) = \begin{pmatrix} \mathring{\mathcal{D}}^*(\boldsymbol{\rho})_{ij} - 2\mathring{K}_{\{i}{}^k\eta_{j\}k} + \mathring{K}(\eta_{ij} - \frac{1}{3}\eta_k{}^k)\mathring{h}_{ij} + \mathring{\epsilon}_{kl(i}\mathring{\mathring{S}}_{j)}{}^l\varsigma^k \\ \mathring{\epsilon}_{ijk}\mathring{D}_l\rho^{jkl} + \mathring{\epsilon}_{jkl}D^l\rho^{jk}{}_i + \mathring{\epsilon}_{ijl}\mathring{D}^k(\mathring{K}_k{}^l\varsigma^j) + \mathring{\epsilon}_{jkl}\mathring{D}^l(\mathring{K}_i{}^k\varsigma^j) \\ \mathring{\delta} \circ \mathring{L}(\boldsymbol{\varsigma})_i \\ \frac{1}{2}\mathring{\Delta}_L\eta_{ij} + \frac{1}{3}\mathring{D}_i\mathring{D}_j\eta_k{}^k - \frac{1}{6}\mathring{h}_{ij}\mathring{\Delta}\eta_k{}^k + \mathscr{G}(\boldsymbol{\eta}, \boldsymbol{\varsigma}, \boldsymbol{\rho}, \boldsymbol{\rho})_{ij} \end{pmatrix}, \qquad (6.2.8)$$

where the index contractions and the braces (representing the symmetric tracefree part) are with respect to $\mathring{\boldsymbol{h}}$, and $\mathscr{G}(\cdot)$ is the linear operator determined by

$$\langle \mathscr{G}(\boldsymbol{\eta}, \boldsymbol{\varsigma}, \boldsymbol{\rho}, \boldsymbol{\rho}), \boldsymbol{\gamma} \rangle = \langle \boldsymbol{\rho}, \mathscr{F}(\boldsymbol{\gamma}) \rangle + \langle \boldsymbol{\varsigma}, \mathscr{F}_3(\boldsymbol{\gamma}) \rangle + \langle \boldsymbol{\eta}, \mathscr{F}_5(\boldsymbol{\gamma}) \rangle,$$

namely

$$\begin{aligned}\mathscr{G}(\boldsymbol{\eta},\boldsymbol{\varsigma},\boldsymbol{\rho},\boldsymbol{\rho})_{ij} \equiv & -\tfrac{2}{3}\lambda\eta_{ij} + \eta^{kl}(\mathring{K}_{ik}\mathring{K}_{jl} - \mathring{K}_{ij}\mathring{K}_{kl}) + \tfrac{1}{2}\mathcal{L}_{\boldsymbol{\varsigma}}\mathring{S}_{ij} - \tfrac{1}{2}\mathcal{L}_{\boldsymbol{\rho}}\mathring{K}_{ij} \\ & + (\mathring{\epsilon}_{km\{i}\mathring{K}^{lm}\mathring{\bar{S}}_{j\}l} + \mathring{\epsilon}_{klm}\mathring{K}_{\{i}{}^{l}\mathring{\bar{S}}_{j\}}{}^{m} - \mathring{\epsilon}_{km\{i}\mathring{K}_{j\}}{}^{l}\mathring{\bar{S}}_{l}{}^{m})\varsigma^{k} \\ & + \tfrac{1}{6}\mathring{S}_{ij}(\mathring{\delta}(\boldsymbol{\varsigma}) - 2\eta_{k}{}^{k}) + \tfrac{1}{4}(\mathring{K}_{kl}\mathring{L}(\boldsymbol{\rho})^{lk} - \mathring{S}_{kl}\mathring{L}(\boldsymbol{\varsigma})^{lk} + 2\rho^{k}\mathring{\delta}(\boldsymbol{K})_{k})\mathring{h}_{ij} + \tfrac{1}{2}\mathring{K}_{ij}\mathring{\delta}(\boldsymbol{\rho}) \\ & - \tfrac{1}{2}\mathring{\epsilon}_{lm(i}\mathring{K}^{kl}\mathring{D}_{|k|}\rho_{j)}{}^{m} + \tfrac{1}{2}\mathring{\epsilon}_{km(i}\mathring{K}_{j)}{}^{k}\mathring{\delta}(\boldsymbol{\rho})^{m} + \tfrac{1}{2}\mathring{\epsilon}_{klm}K_{(i}{}^{k}\mathring{D}^{m}\rho_{j)}{}^{l} \\ & + \tfrac{3}{2}\rho_{\{i}{}^{k}\mathring{\bar{S}}_{j\}k} + \tfrac{1}{2}\rho_{(i}{}^{k}\mathring{\epsilon}_{j)km}\mathring{\delta}(\mathring{\boldsymbol{K}})^{m} - \tfrac{1}{2}\rho_{k}{}^{l}\mathring{\epsilon}_{lm(i}\mathring{D}^{k}K_{j)}{}^{m}.\end{aligned}$$

The above expression has been simplified using the ECEs satisfied by the background fields $\mathring{K}_{ij}$,$\mathring{\bar{S}}_{ij}$,$\mathring{S}_{ij}$ and $\mathring{h}_{ij}$. Here, $\rho_i \in \Lambda^1(\mathcal{S})$, $\rho_{ij} \in \mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})$ are the quantities arising in the Jacobi decomposition of ρ_{ijk}:

$$\rho_{ijk} = \tfrac{1}{2}(\mathring{\epsilon}_{ij}{}^{l}\rho_{kl} + \rho_i\mathring{h}_{jk} - \rho_j\mathring{h}_{ik}).$$

Substituting the decomposition into the first two components of $(D_v\widetilde{\Psi})^* = 0$, one obtains

$$\begin{aligned}0 &= \mathring{\mathcal{R}}(\boldsymbol{\rho})_{ij} + \tfrac{1}{2}\mathring{L}(\boldsymbol{\rho})_{ij} - 2\mathring{K}_{k\{i}\eta_{j\}}{}^{k} + \mathring{K}(\eta_{ij} - \tfrac{1}{3}\eta_{k}{}^{k}\mathring{h}_{ij}) + \varsigma^{k}\mathring{\epsilon}_{kl(i}\mathring{\bar{S}}_{j)}{}^{l}, \\ 0 &= 2\mathring{D}^{j}\rho_{ij} + \varsigma^{j}\mathring{\epsilon}_{ijl}\mathring{\delta}(\mathring{\boldsymbol{K}})^{l} - \mathring{\epsilon}_{jkl}\mathring{K}_{i}{}^{l}\mathring{D}^{k}\varsigma^{j} + \mathring{\epsilon}_{ijl}\mathring{K}_{k}{}^{l}\mathring{D}^{k}\varsigma^{j} + \varsigma^{j}\mathring{\epsilon}_{jkl}\mathring{D}^{l}\mathring{K}_{i}{}^{k}.\end{aligned}$$

Setting $\varsigma_i = 0,\ \rho_{ij} = \bar{\eta}_{ij} = 0$, the latter trivialises and the former reduces to

$$0 = \tfrac{1}{2}\mathring{L}(\boldsymbol{\rho})_{ij} - \tfrac{2}{3}(\mathring{K}_{ij} - \tfrac{1}{3}\mathring{K}\mathring{h}_{ij})\eta. \tag{6.2.9}$$

On the other hand, substituting into the last component of $(D_v\widetilde{\Psi})^* = 0$, one obtains

$$\begin{aligned}0 &= -\tfrac{1}{3}\eta\mathring{S}_{ij} + \tfrac{1}{3}\eta\mathring{K}_{i}{}^{k}\mathring{K}_{jk} - \tfrac{1}{3}\eta\mathring{K}\mathring{K}_{ij} - \tfrac{2}{9}\lambda\eta\mathring{h}_{ij} + \tfrac{1}{3}\mathring{D}_{j}\mathring{D}_{i}\eta + \tfrac{1}{2}\mathring{K}_{ij}\mathring{D}^{k}\rho_{k} \\ &\quad - \tfrac{1}{3}\mathring{h}_{ij}\mathring{\Delta}\eta + \tfrac{1}{2}\rho^{k}\mathring{h}_{ij}\mathring{\delta}(\mathring{\boldsymbol{K}})_{k} + \tfrac{1}{2}\mathring{K}_{kl}(\mathring{D}^{l}\rho^{k})\mathring{h}_{ij} - \tfrac{1}{2}\mathcal{L}_{\rho}\mathring{K}_{ij} \\ &= \tfrac{1}{3}\eta(2\mathring{K}_{i}{}^{k}\mathring{K}_{jk} - 2\mathring{K}_{ij}\mathring{K} - \mathring{r}_{ij}) + \tfrac{1}{3}\mathring{D}_{i}\mathring{D}_{j}\eta - \tfrac{1}{3}\mathring{h}_{ij}\mathring{\Delta}\eta + \tfrac{1}{2}\mathring{K}_{ij}\mathring{D}^{k}\rho_{k} \\ &\quad + \tfrac{1}{2}\rho^{k}\mathring{h}_{ij}\mathring{\delta}(\mathring{\boldsymbol{K}})_{k} + \tfrac{1}{2}\mathring{K}_{kl}(\mathring{D}^{l}\rho^{k})\mathring{h}_{ij} - \tfrac{1}{2}\mathcal{L}_{\rho}\mathring{K}_{ij}\end{aligned} \tag{6.2.10}$$

where the second equality uses the background Gauss–Codazzi equation $\mathring{V}_{ij} = 0$ to substitute for $\mathring{S}_{ij}$. Together, equations (6.2.9) and (6.2.10) are precisely

$$\begin{pmatrix} \Pi_{\mathring{\boldsymbol{h}}} & 0 \\ 0 & 1 \end{pmatrix} D\Phi^* \begin{pmatrix} \tfrac{1}{3}\eta \\ \boldsymbol{\rho} \end{pmatrix} = 0 \tag{6.2.11}$$

—see (6.2.7). Now, since $D\Phi^* = 0$ is equivalent to the KID equations, a KID set $(\mathcal{N}, Y^i)$ gives rise to an element $(\eta,\rho^i) = (3\mathcal{N}, Y^i)$ in the kernel of $(D_v\widetilde{\Psi})^*$ and hence an obstruction to solving $\widetilde{\Psi} = 0$. To see why this should be the case, note first that the linearisations of $\mathrm{tr}_{\boldsymbol{h}}\mathrm{Ric}^{L,\boldsymbol{X}}$ and $r[\boldsymbol{h}]$ agree:

$$\begin{aligned}\frac{d}{d\tau}r[\boldsymbol{h}+\tau\boldsymbol{\gamma}]\bigg|_{\tau=0} &= \mathrm{tr}_{\mathring{\boldsymbol{h}}}(D\mathrm{Ric}(\boldsymbol{\gamma})) - \langle\boldsymbol{\gamma},\mathrm{Ric}[\mathring{\boldsymbol{h}}]\rangle_{\mathring{\boldsymbol{h}}} \\ &= \mathrm{tr}_{\mathring{\boldsymbol{h}}}(D\mathrm{Ric}^{L}(\boldsymbol{\gamma})) + \tfrac{1}{2}\mathrm{tr}_{\mathring{\boldsymbol{h}}}(\mathring{L}\circ\mathring{B}(\boldsymbol{\gamma})) - \langle\boldsymbol{\gamma},\mathrm{Ric}[\mathring{\boldsymbol{h}}]\rangle_{\mathring{\boldsymbol{h}}} \\ &= \mathrm{tr}_{\mathring{\boldsymbol{h}}}(D\mathrm{Ric}^{L,\boldsymbol{X}}(\boldsymbol{\gamma})) - \langle\boldsymbol{\gamma},\mathrm{Ric}[\mathring{\boldsymbol{h}}]\rangle_{\mathring{\boldsymbol{h}}} \\ &= \frac{d}{d\tau}r^{L}[\boldsymbol{h}+\tau\boldsymbol{\gamma}]\bigg|_{\tau=0},\end{aligned}$$

where we are using the fact that $Q(\mathring{\boldsymbol{h}})_i = 0$ so that $\mathrm{Ric}^L[\mathring{\boldsymbol{h}}]_{ij} = \mathrm{Ric}[\mathring{\boldsymbol{h}}]_{ij}$ and that the image $\mathring{L}$ is $\mathring{\boldsymbol{h}}-$tracefree. Hence, when one linearises the gauge-reduced equation $\widetilde{V}_{ij} = 0$ and traces, one obtains precisely the linearised Hamiltonian constraint. On the other hand, $J_{ij}{}^j = 0$ is precisely the momentum constraint. The appearance of the operator Π in equation (6.2.11) occurs because the determined field σ_{ij} lies in the space $\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})$ and so the adjoint must map into this space rather than into $\mathscr{S}^2(\mathcal{S})$.

In the case of umbilical data, the first component of equation (6.2.11) reduces to the conformal Killing equation on $(\mathcal{S},\mathring{\boldsymbol{h}})$, and we recover conformal Killing fields as obstructions, as in Chapter 4.

Remark 42. Of course, it would be nice if one could obtain a precise geometric characterisation of the obstructions i.e. of the cokernel of $D_v\widetilde{\Psi}$. By analogy to the problem of linearisation stability, it reasonable to suspect that the elements of the kernel may correspond to the projections of some geometric object on the ambient spacetime development. This is postponed to future work.

6.3 Constructing solutions of the ECEs

In this section, we will discuss the application of the above framework to the construction of solutions of the ECEs. First, we define the following Banach spaces, for $l \in \mathbb{N}$,

$$
\begin{aligned}
\mathcal{X}^l &\equiv H^{l+2}(\mathscr{C}(\mathcal{S})) \times H^{l+2}(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}})) \times H^{l+2}(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}})),\\
\mathcal{Y}^l &\equiv H^{l+2}(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})) \times H^{l+2}(\Lambda^1(\mathcal{S})) \times H^{l+2}(\Lambda^1(\mathcal{S})) \times H^{l+3}(\mathscr{S}^2(\mathcal{S})),\\
\mathcal{Z}^l &\equiv H^{l+1}(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})) \times H^{l+1}(\Lambda^1(\mathcal{S})) \times H^{l}(\Lambda^1(\mathcal{S})) \times H^{l+1}(\mathscr{S}^2(\mathcal{S})).
\end{aligned}
$$

The exponents in the definition of $\mathcal{Y}^l$ and $\mathcal{Z}^l$ are equal to $l+t_\nu$, $l-s_\mu$, respectively, where $s_\mu,\ t_\nu$ are the Douglis–Nirenberg weights identified in the proof of Proposition 18. In order to establish surjectivity of $D_v\widetilde{\Psi}$, we shall have to consider the kernel of the adjoint $(D_v\widetilde{\Psi})^*$ (which is Douglis–Nirenberg elliptic with the same weights as $D_v\widetilde{\Psi}$) acting on $\mathcal{Z}^l$. To be able to apply elliptic regularity, we require $l \geq 2$ in order that

$$
l+1 \geq t_1 = 2, \qquad l+1 \geq t_2 = 2, \qquad l \geq t_3 = 2, \qquad l+1 \geq t_4 = 3.
$$

—the left-hand-sides of the inequalities being the exponents appearing in $\mathcal{Z}^l$. In other words, given $l \geq 2$, ker $(D_v\widetilde{\Psi})^* \cap \mathcal{Z}^l \subset C^\infty$ —see also Remark 41. For what follows, define the map

$$
\omega : (\phi, \bar{\boldsymbol{T}}, \boldsymbol{T}, \boldsymbol{\chi}, \bar{\boldsymbol{X}}, \boldsymbol{X}) \mapsto \begin{pmatrix} \chi_{ij} + \frac{1}{3}\phi \mathring{h}_{ij} \\ \bar{\boldsymbol{S}}(\bar{\boldsymbol{X}}, \bar{\boldsymbol{T}})_{ij} \\ \boldsymbol{S}(\boldsymbol{X}, \boldsymbol{T})_{ij} \end{pmatrix},
$$

as in Chapters 4 and 5. Then we have the following

Proposition 19. Let $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$ be a smooth closed initial data set. The map $\widetilde{\Psi} : \mathcal{X}^l \times \mathcal{Y}^l \to \mathcal{Z}^l$ is continuous. Fixing $l \geq 2$, ker $D_v\widetilde{\Psi}$ and ker $(D_v\widetilde{\Psi})^*$ are finite-dimensional and consist of smooth sections. If both kernels trivialise[3] then there exist open neighbourhoods $\mathcal{U} \subseteq \mathcal{X}^l$, $\mathcal{V} \subseteq \mathcal{Y}^l$ of $(\mathring{K}, \mathring{\bar{\boldsymbol{T}}}, \mathring{\boldsymbol{T}})$ and $(\mathring{\boldsymbol{\chi}}, \mathring{\bar{\boldsymbol{\xi}}}, \mathring{\boldsymbol{\xi}}, \mathring{\boldsymbol{h}})$ and a map $\nu : \mathcal{U} \to \mathcal{V}$ such that $w(\nu(u), u)$ is a solution for the ECEs for each $u \in \mathcal{U}$.

[3] In particular, $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$ must admit no KID sets, by the previous discussion.

Proof. The fact that $\widetilde{\Psi}$ is a continuous map from $\mathcal{X}^l \times \mathcal{Y}^l$ to $\mathcal{Z}^l$ is easily verified using the Schauder ring property —see Section 2.3. Finite dimensionality of ker $D_v\widetilde{\Psi}$ and ker $(D_v\widetilde{\Psi})^*$ follows from Theorem 5 applied to $P = D_v\widetilde{\Psi}$ and $P = (D_v\widetilde{\Psi})^*$, and smoothness (when $l \geq 2$) is a consequence of elliptic regularity, as described above. If both kernels trivialise then $D_v\widetilde{\Psi} : \mathcal{Y}^l \to \mathcal{Z}^l$ is an isomorphism by the Fredholm alternative —see Theorem 5. The Implicit Function Theorem guarantees the existence of $\mathcal{U}, \mathcal{W}$ and a map ν such that $\widetilde{\Psi}(\nu(u); u) = 0$ for each $u \in \mathcal{U}$. It remains to show that the candidate solutions $w(u) \equiv w(u, \nu(u))$ indeed satisfy the ECEs. First note the zero quantities constructed from $w(u)$ satisfy (by definition of $\widetilde{\Psi}$)

$$J_{ijk} = 0, \qquad \Lambda_i = 0, \qquad V_{ij} = \tfrac{1}{2} L(\boldsymbol{Q}^{\boldsymbol{X}})_{ij}. \tag{6.3.1}$$

Therefore, all that remains to be shown is that

$$\bar{\Lambda}_i = 0, \qquad L(Q^{\boldsymbol{X}})_{ij} = 0.$$

First consider the latter. Substituting (6.3.1) into the integrability condition (4.1.11b), we find that

$$\begin{aligned} 0 = -2\langle Q^{\boldsymbol{X}}, B \circ L(Q^{\boldsymbol{X}})\rangle_{L^2} &= -2\langle Q^{\boldsymbol{X}}, \delta \circ L(Q^{\boldsymbol{X}}) - \tfrac{1}{2} d(\mathrm{tr}_{\boldsymbol{h}} L(Q^{\boldsymbol{X}}))\rangle_{L^2} \\ &= -2\langle Q^{\boldsymbol{X}}, \delta \circ L(Q^{\boldsymbol{X}})\rangle_{L^2} \\ &= \|L(Q^{\boldsymbol{X}})\|^2_{L^2}. \end{aligned}$$

Since $\mathcal{S}$ is closed, we can integrate by parts to find that $L(Q^{\boldsymbol{X}})_{ij} = 0$, as required. On the other hand, $\bar{\Lambda}_i = 0$ follows automatically by virtue of the remaining integrability condition

$$\bar{\Lambda}_l + \tfrac{1}{2}\epsilon_{ijk} D^k J^{ij}{}_l = 0.$$

Hence, the candidate solutions $w(u)$ indeed solve the ECEs. □

Remark 43. The argument to show that $L(Q^{\boldsymbol{X}})_{ij} = 0$ can also be applied in the asymptotically-Euclidean setting provided one chooses the functional spaces in such a way that the fields are of the appropriate decay to permit integration by parts.

6.3.1 Application to time symmetric background initial data sets

Let us return to the case of time symmetric background initial data set. On such a background, the linearised auxiliary equations, $D_v\widetilde{\Psi} = 0$ (see (6.2.1)), reduce to

$$\mathring{\mathcal{D}}(\boldsymbol{\chi})_{ijk} - \mathring{\epsilon}_{ij}{}^l \mathring{L}(\bar{\boldsymbol{\xi}})_{kl} = 0, \tag{6.3.2a}$$

$$\mathring{\delta} \circ \mathring{L}(\boldsymbol{\xi})_i - \mathring{S}_i{}^j \mathring{B}(\boldsymbol{\gamma})_j - \tfrac{1}{2}\mathring{S}^{jk}\mathring{D}_i\gamma_{jk} - \gamma^{jk}\mathring{D}_k\mathring{S}_{ij} = 0, \tag{6.3.2b}$$

$$\tfrac{1}{2}\mathring{P}_L\gamma_{ij} - \tfrac{1}{3}(\mathring{S}^{kl}\gamma_{kl})\mathring{h}_{ij} + \tfrac{1}{3}\mathring{\delta} \circ \mathring{B}(\boldsymbol{\gamma})\mathring{h}_{ij} = 0. \tag{6.3.2c}$$

Recall that equation (6.3.2a) is equivalent to $\mathring{\mathcal{P}}^{(0)}(\boldsymbol{\chi}, \bar{\boldsymbol{\xi}}) = 0$. The adjoint system, $(D_v\widetilde{\Psi})^* = 0$ (see (6.2.8)), reduces to

$$\mathring{\mathcal{R}}(\boldsymbol{\rho})_{ij} + \tfrac{1}{2}\mathring{L}(\boldsymbol{\varrho})_{ij} = 0, \tag{6.3.3a}$$
$$2\mathring{\delta}(\boldsymbol{\rho})_i = 0, \tag{6.3.3b}$$
$$\mathring{\delta} \circ \mathring{L}(\boldsymbol{\varsigma})_i = 0, \tag{6.3.3c}$$
$$\tfrac{1}{2}\mathring{P}_L\eta_{ij} + \tfrac{1}{6}(2\mathring{D}_i\mathring{D}_j - \mathring{h}_{ij}\mathring{\Delta})\eta + \tfrac{1}{2}\mathcal{L}_{\boldsymbol{\varsigma}}\mathring{S}_{ij} + \tfrac{1}{6}(\mathring{\delta}(\boldsymbol{\varsigma}) - 2\eta)\mathring{S}_{ij} - \tfrac{1}{4}\mathring{S}_{kl}\mathring{L}(\boldsymbol{\varsigma})^{kl}\mathring{h}_{ij} = 0. \tag{6.3.3d}$$

We have the following improvement on Theorem 3 of Chapter 5:

Theorem 6. Fix $l \geq 2$. Suppose $(\mathcal{S}, \mathring{\boldsymbol{h}})$ is a constant scalar curvature Riemannian manifold ($\mathring{r} = 2\lambda$) satisfying

$$\begin{aligned}
&(A1): \quad \mathring{\Delta} + \lambda \quad \text{injective on} \quad C^\infty(\mathcal{S}),\\
&(A2): \quad \mathring{P}_L \quad \text{injective on} \quad \Gamma(\mathscr{S}^2_0(\mathcal{S}, \mathring{\boldsymbol{h}}))),\\
&(A3): \quad \mathring{\mathcal{P}}^{(0)} \quad \text{injective on} \quad \Gamma(\mathscr{S}^2_0(\mathcal{S}, \mathring{\boldsymbol{h}})) \oplus \Gamma(\Lambda^1(\mathcal{S})).
\end{aligned}$$

Then $D_v\widetilde{\Psi} : \mathcal{Y}^l \to \mathcal{Z}^l$ is an isomorphism of Banach spaces. There exists an open neighbourhood $\mathcal{U} \subseteq \mathcal{X}^l$ of $(\mathbf{0}, \mathbf{0}, \mathring{\boldsymbol{S}})$, an open neighbourhood $\mathcal{W} \subseteq \mathcal{Y}^l$ of $(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathring{\boldsymbol{h}})$ and a smooth map $\nu : \mathcal{U} \to \mathcal{W}$ such that, defining

$$u \equiv (\phi, \bar{\boldsymbol{T}}, \boldsymbol{T}), \qquad \nu(u) \equiv \big(\boldsymbol{\chi}(u), \bar{\boldsymbol{X}}(u), \boldsymbol{X}(u), \boldsymbol{h}(u)\big),$$

the following assertions hold:

i) for each $(\phi, \bar{\boldsymbol{T}}, \boldsymbol{T}) \in \mathcal{U}$,

$$w(u) \equiv \big(\boldsymbol{\chi}(u) + \tfrac{1}{3}\phi\mathring{\boldsymbol{h}},\ \bar{\boldsymbol{S}}(\bar{\boldsymbol{X}}(u), \bar{\boldsymbol{T}}),\ \boldsymbol{S}(\boldsymbol{X}(u), \boldsymbol{T}),\ \boldsymbol{h}(u)\big)$$

is a solution to the extended constraint equations with cosmological constant λ;

ii) the map $u \mapsto w(u)$ is injective if we restrict the free datum ϕ to the sub-Banach space $\bar{H}^{l+2}(\mathscr{C}(\mathcal{S}))$.

Proof. **Injectivity of $D_v\widetilde{\Psi}$:** First note that any solution to $D_v\widetilde{\Psi} = 0$ must be smooth by elliptic regularity. Equation (6.3.2a) immediately implies that $\chi_{ij} = 0$ and $\bar{\xi}_i = 0$, condition (A3). Taking the tracefree part of (6.3.2c), one obtains $\mathring{P}_L\bar{\gamma}_{ij} = 0$ and so (A2) implies $\bar{\gamma}_{ij} = 0$. Substituting back into (6.3.2c) and recalling that $\mathring{S}_i{}^i = 0$, we find

$$(\mathring{\Delta} + \lambda)\gamma = 0$$

and so condition (A1) implies $\gamma = 0$. Hence, $\gamma_{ij} = 0$. Substituting into (6.3.2b), we find $\mathring{\delta} \circ \mathring{L}(\boldsymbol{\xi}) = 0$, implying $\xi_i = 0$ by condition (A3) —recall that $\ker \mathring{L} \subset \ker \mathring{\mathcal{P}}^{(0)} = \{0\}$. Hence, $D_v\widetilde{\Psi}$ is injective.

Surjectivity of $D_v\widetilde{\Psi}$: By the Fredholm alternative it suffices to verify injectivity of the adjoint map. Note that any solution to $(D_v\widetilde{\Psi})^* = 0$ is again automatically smooth by elliptic regularity. The equations (6.3.3a)–(6.3.3b) are precisely $\mathring{\mathcal{P}}^{(0)}(\boldsymbol{\rho}, \boldsymbol{\rho})$, so condition (A3) implies $\rho_{ij} = 0,\ \rho_i = 0$. Since (A3) implies non-existence of conformal Killing vectors, equation (6.3.3c) implies that $\varsigma_i = 0$. Substituting into (6.3.3d),

$$\tfrac{1}{2}\mathring{\Delta}_L\eta_{ij} - \tfrac{2}{3}\lambda\eta_{ij} + \tfrac{1}{6}(-\mathring{h}_{ij}\mathring{\Delta}\eta + 2\mathring{D}_i\mathring{D}_j\eta) = 0.$$

Taking the trace one obtains

$$(\mathring{\Delta}+\lambda)\eta = 0,$$

implying that $\eta = 0$ by condition (A1). Substituting back in we find that $\mathring{P}_L\bar{\eta}_{ij} = 0$, implying $\bar{\eta}_{ij} = 0$ by condition (A2), and hence $\eta_{ij} = 0$. Hence, $D_v\widetilde{\Psi}$ is surjective.

Applying Proposition 19 then proves statement *i*). For statement *ii*) we need only show that $D_u\widetilde{\Psi}$ is injective. It is straightforward to compute that, for a time symmetric background,

$$D_u\widetilde{\Psi}(\breve{\phi},\breve{\breve{\boldsymbol{T}}},\breve{\boldsymbol{T}}) = \begin{pmatrix} \frac{1}{3}\mathring{\mathcal{D}}(\breve{\phi}\mathring{\boldsymbol{h}})_{ijk} - \mathring{\epsilon}_{ij}{}^{l}\breve{\breve{T}}_{kl} \\ \mathring{\delta}(\breve{\boldsymbol{T}})_i \\ -\breve{T}_{ij} \end{pmatrix}.$$

Hence, $D_u\widetilde{\Psi}(\breve{\phi},\breve{\breve{\boldsymbol{T}}},\breve{\boldsymbol{T}}) = 0$ immediately implies that $\breve{T}_{ij} = 0$. Decomposing the first equation and using the fact that $\mathring{\mathcal{R}}(f\mathring{\boldsymbol{h}})_{ij} = 0$ for all scalar functions f, one finds that $\breve{\breve{T}}_{ij} = 0$ and $d\breve{\phi}_i = 0$. The latter implies that $\breve{\phi}$ is constant, and therefore if we restrict to $\bar{H}^{l+2}(\mathscr{C}(\mathcal{S}))$ then $\breve{\phi} = 0$ and $D_u\widetilde{\Psi}$ is injective. □

Remark 44. For umbilical initial data, the KID equations (6.2.6a)–(6.2.6b) can be shown to reduce to

$$\tfrac{1}{3}\mathcal{N}Kh_{ij} + D_{(i}Y_{j)} = 0, \tag{6.3.4a}$$
$$D_iD_j\mathcal{N} - \mathcal{N}(r_{ij} - \tfrac{1}{2}rh_{ij}) = 0. \tag{6.3.4b}$$

The latter is known in the literature as the *static potential equation*, and solutions $\mathcal{N}$ referred to as *static potentials*. In the time-symmetric sub-case, $K = 0$, the existence of a static potential would imply a non-trivial KID set (take $Y_i = 0$, for example) and therefore a non-trivial element of $\ker\,(D_v\widetilde{\Psi})^*$. In the case $K \neq 0$ it is not so clear whether a static potential $\mathcal{N}$ can be completed to a KID set, but one still has

$$\begin{pmatrix} \mathring{\Pi} & 0 \\ 0 & 1 \end{pmatrix} D\Phi^* \begin{pmatrix} \mathcal{N} \\ 0 \end{pmatrix} = 0,$$

since the tracefree part of (6.3.4a) decouples from (6.3.4b), and this gives rise to non-trivial elements in $\ker\,(D_v\widetilde{\Psi})^*$. Note also that, taking the trace of the static potential equation, one obtains

$$(\Delta + \tfrac{1}{2}r)\mathcal{N} = 0.$$

Hence, condition (A1) implies the non-existence of static potentials, but clearly imposes a much stronger restriction on $(\mathcal{S},\mathring{\boldsymbol{h}})$. It would be interesting to see whether (A1) in Theorem 6 can be weakened to the requirement that $(\mathcal{S},\mathring{\boldsymbol{h}})$ admit no non-trivial static potentials.

The above theorem is an improvement on Theorem 3 in that we do not impose any restrictions on the kernels of $\mathring{\Delta}_Y$ and $\mathring{\mathcal{P}}^{(1)}$.

6.4 Concluding remarks

In this chapter an alternative auxiliary extended constraint map was presented which streamlines the approaches of the previous chapters, in the sense of simplifying the analysis of the linearised equations (for time symmetric background initial data) and rendering the sufficiency argument almost trivial. The two new additions are the use of a new gauge-reduction procedure and the use of an inbuilt mixed-order ellipticity of the ECEs. As an additional pay-off, it was shown that KID sets naturally arise as obstructions to solving the auxiliary system, which is desirable given the role that KID sets play in the problem of linearisation stability. The proposed method was then applied to time symmetric background initial data sets and the conditions (A1)–(A3) were identified as being sufficient conditions for the method's implementation, thereby improving on Theorem 3 of Chapter 5.

So far we have been unable to give a natural geometric interpretation of the full cokernel, of which the KID sets are only part. It is possible that the obstructions may be given a more geometric characterisation by identifying them with components of some tensor field on the full spacetime manifold, in a way that is analogous to the identification of KID sets with the lapse-shift components of spacetime Killing vectors —see Section 6.2.2. It is possible that the linearised auxiliary system would be further simplified by using a gauge-reduction which is also adapted to the extrinsic curvature perturbation (in addition to the perturbation of the electric part). At present it is not clear how to do this, particularly in view of the fact that one expects the "correct" approach to leave the linearised scalar curvature unchanged in order that KID sets continue to feature as (potential) obstructions to integrability. However, such a modification may be necessary if the method is to be applied to non-time symmetric background data.

Another direction of study would involve generalising the method to non-compact $\mathcal{S}$ —e.g. asymptotically-Euclidean, or hyperboloidal, initial data. To do so, one would first have to make use of the elliptic machinery for Douglis–Nirenberg systems on non-compact domains, which can be found in [79] for instance. As discussed in Remark 39, certain aspects of the streamlined method presented in this chapter render it unsuitable for generalisation to the full CCEs, suggesting that an approach more in keeping with that of Chapters 4 and 5 is required.

Chapter 7

Extending the Friedrich–Butscher method to the full CCEs

In this chapter we return to the full Conformal Constraint Equations (CCEs), and the problem of generalising the Friedrich–Butscher method to this context. In Section 7.1, we will describe an elliptic reduction of the CCEs, involving a specification of the free and determined fields that is motivated by the Friedrich–Butscher method. The new feature that arises here is that certain parts of the free data (identified in Section 7.1) are "unphysical" in the sense that they should be thought of as fixing the conformal gauge freedom inherent to solutions of the CCEs —see Section 3.2.2. In Section 7.2, I will describe an application to the construction of non-linear perturbations of the CCEs around the hyperbolic, conformally-rigid background geometries considered in Chapter 4, thought of now as a solution to the full CCEs with trivial conformal factor $\mathring{\Omega} \equiv 1$. It will be important to note that the hyperbolic manifolds considered earlier, and which we take as defining our background intrinsic geometry, have vanishing first Betti number —see Remark 14. Hence, by Hodge's Theorem, the background manifolds admit no non-trivial harmonic 1−forms —i.e. $\ker \mathring{\Delta}_H = \{\mathbf{0}\}$— a fact that will be used explicitly both in the construction of candidate solutions and in the sufficiency argument.

7.1 An elliptic reduction of the CCEs

For ease of reference, let us recall here the CCEs and their integrability conditions from Chapter 3. The CCEs on $\mathcal{S}$ are given by

$$P_{ij} = 0, \quad Z_i = 0, \quad W_i, \quad X_{ij} = 0, \quad Y_{ijk} = 0, \quad X_{ijk} = 0, \quad \Lambda^*_i = 0, \quad \Lambda_i = 0, \quad A = 0,$$

where the zero quantities are defined as follows

$$P_{ij} \equiv D_i D_j \Omega - \sigma K_{ij} - s h_{ij} + \Omega L_{ij}, \tag{7.1.1a}$$
$$Z_i \equiv D_i \sigma - K_i{}^k D_k \Omega + \Omega L_i, \tag{7.1.1b}$$
$$W_i \equiv D_i s + L_{ij} D^j \Omega - \sigma L_i, \tag{7.1.1c}$$
$$X_{ij} \equiv D_i L_j - D_j L_i + K_j{}^k L_{ik} - K_i{}^k L_{jk} - \epsilon_{ijl} d^{*}_k{}^l D^k \Omega, \tag{7.1.1d}$$
$$X_{ijk} \equiv D_i L_{jk} - D_j L_{ik} - 2K_{k[i} L_{j]} - \epsilon_{ijl} d^{*}_k{}^l \sigma + 2d_{k[i} D_{j]} \Omega - 2d_{l[i} h_{j]k} D^l \Omega, \tag{7.1.1e}$$
$$Y_{ijk} \equiv D_i K_{jk} - D_j K_{ik} - \Omega \epsilon_{ijl} d^{*}_k{}^l + h_{jk} L_i - h_{ik} L_j, \tag{7.1.1f}$$
$$\Lambda^*_i \equiv D^j d^*_{ij} - \epsilon_{ikl} K_j{}^l d^{jk}, \tag{7.1.1g}$$
$$\Lambda_i \equiv D^j d_{ij} + \epsilon_{ikl} K_j{}^l d^{*jk}, \tag{7.1.1h}$$
$$U_{ij} \equiv l_{ij} - L_{ij} - \Omega d_{ij} - K_i{}^k K_{jk} + \tfrac{1}{4} K_{kl} K^{kl} h_{ij} - \tfrac{1}{4} K^2 h_{ij} + K K_{ij}, \tag{7.1.1i}$$
$$A \equiv \lambda - 6\Omega s - 3\sigma^2 + 3 D_i \Omega D^i \Omega, \tag{7.1.1j}$$

and which automatically satisfy the following integrability conditions

$$\begin{aligned}\epsilon_{lik} D^k P_j{}^i = \epsilon_{jli} W^i - \tfrac{1}{2}\Omega \epsilon_{lik} \left(X^{ik}{}_j - 2K_j{}^k Z^i - \tfrac{1}{2}\sigma Y^{ik}{}_j \right) \\ + \epsilon_{jlk} \left(D^i \Omega \right) U_i{}^k + \epsilon_{lik} \left(D^i \Omega \right) U_j{}^k,\end{aligned} \tag{7.1.2a}$$
$$D_i Z_j - D_j Z_i = \Omega X_{ij} + K_{ik} P_j{}^k - K_{jk} P_i{}^k - \left(D^k \Omega \right) Y_{ijk}, \tag{7.1.2b}$$
$$D_i W_j - D_j W_i = L_i Z_j - L_j Z_i - L_{ik} P_j{}^k + L_{jk} P_i{}^k - \sigma X_{ij} + \left(D^k \Omega \right) X_{ijk}, \tag{7.1.2c}$$
$$\epsilon_{ijk} D^k X^{ij} = \epsilon_{ijl} K_k{}^l X^{ijk} - 2 d^*_{ij} P^{ij} - \epsilon_{ijl} L_k{}^l Y^{ijk} - 2 \left(D_i \Omega \right) \Lambda^{*i}, \tag{7.1.2d}$$
$$\begin{aligned}\epsilon_{ijk} D^k X^{ij}{}_l = -2\epsilon_{ljk} d_i{}^k P^{ij} + \epsilon_{ijk} K_l{}^k X^{ij} - 2d^*_{li} Z^i - \epsilon_{ijk} L^i Y^{jk}{}_l \\ + 2\epsilon_{ljk} L_i{}^k U^{ij} - 2\sigma \Lambda^*_l + 2\epsilon_{lij} (D^j \Omega) \Lambda^i,\end{aligned} \tag{7.1.2e}$$
$$\epsilon_{ijk} D^k Y^{ij}{}_l = -2\Omega \Lambda^*_l + \epsilon_{lij} X^{ij} - 2\epsilon_{ljk} K^{ij} U_i{}^k, \tag{7.1.2f}$$
$$D_i U_j{}^i - D_j U^i{}_i = X_j{}^i{}_i - \Omega \Lambda_j + K_{ik} Y_j{}^{ik} - K_{jk} Y^{ik}{}_i - K Y_j{}^i{}_i, \tag{7.1.2g}$$
$$D_i A = -6\Omega W_i - 6\sigma Z_i + 6 (D^j \Omega) P_{ij}. \tag{7.1.2h}$$

Recall that $U_{ij} = 0$ is equivalent to $V_{ij} = 0$, with

$$V_{ij} \equiv U_{ij} + (\mathrm{tr}_{\boldsymbol{h}} \boldsymbol{U}) h_{ij} = r_{ij} - L_{ij} - L h_{ij} - \Omega d_{ij} - K_i{}^k K_{jk} + K K_{ij}.$$

It will be convenient to pass back and forth between U_{ij} and V_{ij} at various points. We will proceed to study the CCEs in a way that is analogous to previous chapters. That is to say, we will construct an auxiliary system of equations with the property that their linearisation (in the direction of the appropriate determined fields) is elliptic. In principle, an application of the IFT (making use of the Fredholm theory of elliptic operators) allows one to construct solutions to the auxiliary equations with given (appropriately identified) free data. Since the method presented in Chapter 6 is unsuitable for the full CCEs (see Remark 39) we instead follow an approach similar to that of Chapter 4. Having constructed candidate solutions, one must again prove sufficiency of the auxiliary system. Again, the argument will rely crucially on the use of the integrability conditions.

7.1.1 Free and determined data

As in the case of the ECEs, our approach to obtaining an elliptic auxiliary system will be to make use of an appropriate ansatz, the components of which will be divided into free data and determined fields. As mentioned in the introduction to this chapter, certain components of the free data should be interpreted as conformal gauge functions —that is to say, they single out a conformal representative from the family of conformally-equivalent family of solutions to the CCEs.

The intrinsic conformal freedom is fixed by prescribing the trace of the tangential-tangential components of the Schouten tensor. On the other hand, the extrinsic conformal freedom will be fixed by prescribing the scalar part of L_i under a *Helmholtz decomposition*. This generalises the procedure of Section 3.3.1 for umbilical initial data, in which the extrinsic conformal gauge freedom was fixed by choosing a conformal representation in which the hypersurface is maximally embedded —i.e. for which $K = 0$.

The ansatz

First let us consider those equations with principal part consisting of the Codazzi operator, $\mathcal{D}$, namely equations (7.1.1f) and (7.1.1e) for the extrinsic curvature, K_{ij}, and the tangential-tangential component of the 4−dimensional Schouten tensor, L_{ij}. Recall that $\mathcal{D}_{\boldsymbol{h}}$ is overdetermined elliptic on $\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$ —see Lemma 4. Accordingly, we need to perform a trace-tracefree decomposition of K_{ij} and L_{ij}. As before, we decompose the extrinsic curvature as

$$K_{ij} = \chi_{ij} + \tfrac{1}{3}(\phi + \mathring{K})\mathring{h}_{ij}$$

with χ_{ij} tracefree with respect to $\mathring{\boldsymbol{h}}$, so that ϕ is the trace of K_{ij} with respect to $\mathring{\boldsymbol{h}}$. On the other hand, for L_{ij}, it will prove more convenient to instead decompose as follows

$$L_{ij} = \tfrac{\mathring{\lambda}}{6}h_{ij} + \theta_{ij} + \tfrac{1}{3}\theta\mathring{h}_{ij}$$

where $\mathring{\lambda}$ denotes the cosmological constant of the background solution.[1] The inclusion of the $(\mathring{\lambda}/6)\boldsymbol{h}$−term in the decomposition guarantees, in particular, that for the background solution we have $\theta = 0$ and $\theta_{ij} = 0$.

The equations $\Lambda_i = \Lambda^*_i = 0$ will again be dealt with by means of the projected York ansatz from earlier —i.e. we write

$$d_{ij} = \Pi_{\boldsymbol{h}}(\mathring{L}(\boldsymbol{u}) + \boldsymbol{\psi})_{ij}, \qquad d^*_{ij} = \Pi_{\boldsymbol{h}}(\mathring{L}(\boldsymbol{u}^*) + \boldsymbol{\psi}^*)_{ij},$$

with $u_i, u^*_i \in \Lambda^1(\mathcal{S})$ and $\psi_{ij}, \psi^*_{ij} \in \mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}})$.

The equation $X_{ij} = 0$ (equation (7.1.1d)) for L_i is of a different form to those considered in previous chapters and therefore requires a different approach. First note that, since $X_{ij} = X_{[ij]}$, $X_{ij} = 0$ is equivalent to vanishing of the zero quantity

$$X_i \equiv -\tfrac{1}{2}\epsilon_i{}^{jk}X_{jk} \sim -\mathrm{curl}(\boldsymbol{L})_i,$$

where "∼" is again being used as a shorthand for equivalence of principal parts. Recall that the curl operator in dimension 3 is not overdetermined elliptic; it is easily checked that[2]

$$\ker \sigma_{\boldsymbol{\xi}}[\mathrm{curl}] = \mathrm{sp}\langle\boldsymbol{\xi}\rangle.$$

[1] As well shall see, the cosmological constant of the constructed solution, denoted λ, is not prescribed a priori but is determined after the fact, once a candidate solution has been constructed.

[2] This is simply the statement of exactness, at the second entry, of de Rham short exact sequence of symbol maps —see Remark 46.

Moreover, the curl operator fails also to be underdetermined elliptic, which follows from the above by virtue of self-adjointness. The co-vector field L_i may be decomposed according to the *Helmholtz decomposition* as follows

$$L_i = (d\varphi)_i + \mathrm{curl}_{\mathring{\boldsymbol{h}}}(\boldsymbol{\varphi})_i,$$

for some $\varphi \in \mathscr{C}(\mathcal{S})$ and some $\varphi_i \in \Lambda^1(\mathcal{S})$. Note that we are again choosing to decompose with respect to the background metric, $\mathring{\boldsymbol{h}}$. Note also that the decomposition is not unique, since we can always add a constant to φ and a gradient term $(df)_i$ to φ_i. In order to fix the decomposition uniquely, we demand that φ integrate to zero with respect to $d\mu_{\mathring{\boldsymbol{h}}}$ (later this will be imposed by requiring that $\varphi \in \bar{H}^l$ for some $l \in \mathbb{N}$). This can always be effected by the addition to a given $\tilde{\varphi}$ the appropriate unique constant. Moreover, we have $d(\mathscr{C}(\mathcal{S})) \subseteq \ker \mathrm{curl}_{\mathring{\boldsymbol{h}}}$ and so we choose φ_i subject to $\mathring{\delta}(\boldsymbol{\varphi}) = 0$. Again, the latter condition is always achievable —given some $\tilde{\varphi}_i$, $\varphi_i \equiv \tilde{\varphi}_i + (df)_i$ satisfies $\mathring{\delta}(\boldsymbol{\varphi}) = 0$ provided f satisfies

$$\mathring{\Delta} f = -\mathring{\delta}(\tilde{\boldsymbol{\varphi}}),$$

and the latter always admits a solution f, unique up to the addition of a constants, by the Fredholm alternative. This gives rise to a unique φ_i satisfying $\mathring{\delta}(\boldsymbol{\varphi}) = 0$. Fixing φ_i in such a way then

$$0 = X_i \sim \epsilon_i{}^{jk} D_j L_k \sim \mathring{\Delta}_H \varphi_i,$$

where recall that $\mathring{\Delta}_H$ denotes the Hodge Laplacian, which is manifestly elliptic when read as an equation for φ_i. The function φ will be treated as a freely-prescribed datum.

Remark 45. Recall from Hodge theory (see [61], for instance) that for a smooth Riemannian manifold, $(\mathcal{S}, \boldsymbol{h})$,

$$\Lambda^1(\mathcal{S}) = \mathrm{Im}\ d \oplus \mathrm{Im}\ \mathrm{curl} \oplus \ker\ \Delta_H.$$

In the following sections, our choice of background metric is such that $\ker\ \mathring{\Delta}_H = \{\boldsymbol{0}\}$ —indeed, the fact that $\mathring{\Delta}_H$ is injective will be used explicitly. Hence, in this context we are justified in describing the formula

$$L_i = (d\varphi)_i + \mathrm{curl}_{\mathring{\boldsymbol{h}}}(\boldsymbol{\varphi})_i$$

as a *decomposition*, rather than simply an ansatz, since all co-vectors L_i can be written in this form.

Remark 46. Although it will not be used here, we note that there is an alternative elliptic reduction of $X_i = 0$, by considering the equation $\mathrm{curl}(\boldsymbol{X})_i = 0$. Note that

$$\mathrm{curl}(\boldsymbol{X})_i \sim -\mathrm{curl}^2(\boldsymbol{L})_i \equiv -(\Delta_H + d \circ \delta) L_i$$

where we are using the fact that $\Delta_H \equiv \mathrm{curl}^2 - d \circ \delta$. The equation $\mathrm{curl}(\boldsymbol{X})_i = 0$ can be made manifestly elliptic if we instead prescribe $\delta(\boldsymbol{L})_i$, taking over the role of φ in the above approach. More precisely, we could expand $\mathrm{curl}(\boldsymbol{X})_i = 0$ and substitute a given choice of gauge function $F : \mathcal{S} \to \mathbb{R}$ for the $\delta(\boldsymbol{L})-$terms to arrive at the following manifestly elliptic auxiliary equation

$$\Delta_H L_i = -(dF)_i + \mathrm{curl}(\mathcal{G}(\Omega, \boldsymbol{K}, \boldsymbol{L}, \boldsymbol{d}^*))_i, \tag{7.1.3}$$

where $\mathcal{G}(\Omega, \boldsymbol{K}, \boldsymbol{L}, \boldsymbol{d}^*)_i \equiv \epsilon_i{}^{jk} K_j{}^l L_{kl} - d_i^{*j}(d\Omega)_j$. Having constructed a solution to the auxiliary equation(s) one would have to show first that $\delta(\boldsymbol{L}) = F$ so that (7.1.3) reduces to $\mathrm{curl}(\boldsymbol{X})_i = 0$, before then arguing that indeed $X_i = 0$, as required. Note that, using the Helmholtz decomposition,

$$\delta(\boldsymbol{L}) = \delta \circ d(\varphi) + \delta \circ \mathrm{curl}(\boldsymbol{\varphi}) = \Delta\varphi.$$

The fields $\delta(\boldsymbol{L})$ and φ therefore encode the same information. It is reasonable to assume that both reduction methods are in a sense equivalent, owing to duality properties of the de Rham complex

$$0 \to \Lambda^0(\mathcal{S}) \xrightarrow{d} \Lambda^1(\mathcal{S}) \xrightarrow{\mathrm{curl}} \Lambda^1(\mathcal{S}) \xrightarrow{\delta} \Lambda^0(\mathcal{S}) \to 0,$$

which underlie both approaches. This will be explored elsewhere.

Remark 47. Note that for umbilical solutions of the CCEs —see Section 3.3.1— we have from $X_{ij} = 0$ that $L_i = -(1/3)dK$, where recall that K is the unphysical mean extrinsic curvature. This special case may be subsumed into the above discussion by choosing $\varphi_i = 0$ and $\varphi = -(1/3)K$. Hence, the interpretation of φ as a conformal gauge-source function is consistent with the derivation of the umbilical CCEs in Section 3.3.1, in which the function K (see Remark 47) played the role of a conformal gauge-source function —we fixed $K = 0$ there so that the unphysical initial data set was maximally embedded.

Interpreting θ and φ as conformal gauge-source functions

We would like to show that the freely-prescribed fields θ, φ can be naturally thought of as fixing the conformal freedom of the CCEs. For simplicity, consider instead the functions L, ϱ_i defined by

$$L = \mathrm{tr}_{\boldsymbol{h}}(\boldsymbol{L}), \qquad L_i = (d\varrho)_i + \mathrm{curl}_{\boldsymbol{h}}(\boldsymbol{\varrho})_i.$$

Note that here we are performing the decomposition of L_i with respect to $\boldsymbol{h}$, rather than the background metric $\mathring{\boldsymbol{h}}$. We will investigate the relationship of L, ϱ_i to the conformal covariance of the CCEs. Accordingly, consider a transformation

$$\Omega \to \acute{\Omega} \equiv \omega\Omega, \qquad \sigma \to \acute{\sigma} \equiv \sigma + \Omega\varsigma$$

with $\omega > 0$ and ς scalar functions —the *intrinsic* and *extrinsic* conformal rescalings.

First note that, tracing $\acute{U}_{ij} = 0$, we obtain

$$r[\acute{\boldsymbol{h}}] = \acute{K}^2 - \|\acute{\boldsymbol{K}}\|^2_{\boldsymbol{h}} + 4\acute{L}$$

and similarly for $U_{ij} = 0$. Now the transformation formula for L_{ij} —see Section 3.2.2— implies that

$$\acute{L} = \omega^{-2}L + \omega^{-2}\varsigma + \tfrac{3}{2}\omega^{-2}\varsigma^2 - \omega^{-3}\Delta_{\boldsymbol{h}}\omega + \tfrac{1}{2}\omega^{-4}\|d\omega\|^2_{\boldsymbol{h}}.$$

Combined with the transformation formula

$$\acute{K}_{ij} = \omega(K_{ij} + \varsigma h_{ij}),$$

we then obtain

$$r[\acute{\boldsymbol{h}}] = \acute{K}^2 - \|\acute{\boldsymbol{K}}\|^2_{\boldsymbol{h}} + 4\acute{L} = \omega^{-2}r[\boldsymbol{h}] - 4\omega^{-3}\Delta_{\boldsymbol{h}}\omega + 2\omega^{-4}\|d\omega\|^2_{\boldsymbol{h}}.$$

Given a change of variable $\omega = \vartheta^2$ one then obtains Yamabe's equation:

$$\left(-\Delta_{\boldsymbol{h}} + \tfrac{1}{8} r[\boldsymbol{h}]\right) \vartheta = \tfrac{1}{8} r[\acute{\boldsymbol{h}}] \vartheta^5.$$

Hence, the function L can be thought of as fixing the intrinsic conformal gauge freedom.

On the other hand,

$$\begin{aligned} \delta_{\acute{\boldsymbol{h}}}(\acute{\boldsymbol{L}}) &= \omega^{-3} \delta_{\boldsymbol{h}}(\omega \acute{\boldsymbol{L}}) \\ &= \omega^{-3} \delta_{\boldsymbol{h}}(\boldsymbol{L}) - \Delta_{\boldsymbol{h}} \varsigma + D^i(\omega^{-2} \acute{K}_{ij}(d\omega)^j)), \end{aligned} \tag{7.1.4}$$

with index raising and lowering with respect to h^{ij}, h_{ij} and where we are using the transformation law for L_i, namely

$$\omega \acute{L}_i = L_i - (d\varsigma)_i + \omega^{-1} \varsigma (d\omega)_i + \omega^{-1} K_{ij} (d\omega)^j = L_i - (d\varsigma)_i + \omega^{-2} \acute{K}_{ij} (d\omega)^j.$$

Rearranging (7.1.4), one obtains the following equation for the function ς

$$\Delta_{\boldsymbol{h}} \varsigma = \delta_{\boldsymbol{h}}(\boldsymbol{L}) - \omega^3 \delta_{\acute{\boldsymbol{h}}}(\acute{\boldsymbol{L}}) + D^i(\omega^{-2} \acute{K}_{ij}(d\omega)^j)).$$

If we take $\omega \equiv 1$, then we obtain[3]

$$\Delta_{\boldsymbol{h}} \varsigma = \delta_{\boldsymbol{h}}(\boldsymbol{L} - \acute{\boldsymbol{L}}). \tag{7.1.5}$$

Given ϱ, $\acute{\varrho}$ one can then compute $\delta_{\boldsymbol{h}}(\boldsymbol{L}) = \Delta_{\boldsymbol{h}} \varrho$, $\delta_{\boldsymbol{h}}(\acute{\boldsymbol{L}}) = \Delta_{\boldsymbol{h}} \acute{\varrho}$. Substituting into (7.1.5), one can solve for the normal conformal rescaling ς (the solution exists by the Fredholm alternative) which effects the required change $\varrho \to \acute{\varrho}$. Hence, ϱ can be thought of as fixing the extrinsic conformal gauge freedom.

Remark 48. Here we only discussed L, ϱ. A similar treatment of the functions θ, φ is somewhat messy, owing to the fact that they are defined with respect to the background metric (as necessitated by the IFT) rather than the determined metric. However, the intuition developed above suggests a natural connection between θ, φ and the intrinsic/extrinsic conformal gauge freedom.

7.1.2 The auxiliary CCE equations

We are now in a position to give the auxiliary CCE system of equations, which we will denote by $\widetilde{\Xi} = 0$. As in ECE case, the auxiliary map is constructed such that its linearisation in the direction of the determined fields, as identified in the previous section, is elliptic. As in the ECE case, the equation involving the Ricci tensor (here $V_{ij} = 0$) will be gauge-reduced by addition of the de Turck term. Similarly, having substituted a Helmholtz decomposed L_i into $X_i = 0$, we need to add an additional term to cancel terms of the form $d \circ \mathring{\delta}(\breve{\boldsymbol{\varphi}})_i$ arising from the composition curl $\circ$ curl. In

[3]We are free to perform the intrinsic conformal rescaling first, holding σ fixed (i.e. setting $\varsigma = 0$) and then perform a subsequent extrinsic conformal rescaling holding (the new) Ω fixed (i.e. taking $\omega = 1$).

other words, we define the following reduced zero quantity

$$\begin{aligned}
\widetilde{X}_i &\equiv -\tfrac{1}{2}X^{jk}\epsilon_{ijk} + D_i\mathring{\delta}(\boldsymbol{\varphi}) && (7.1.6)\\
&= -\tfrac{1}{2}X^{jk}\epsilon_{ijk} + D_i\delta_{\boldsymbol{h}}(\boldsymbol{\varphi}) + D_i\left(\mathring{\delta}(\boldsymbol{\varphi}) - \delta_{\boldsymbol{h}}(\boldsymbol{\varphi})\right)\\
&= \epsilon_{ikl}K^{jk}L_j{}^l + D_iD_j\varphi^j - D_jD_i\varphi^j + d^*_{ij}D^j\Omega - \Delta\varphi_i + D_i(\mathring{\delta}(\boldsymbol{\varphi}) - \delta_{\boldsymbol{h}}(\boldsymbol{\varphi}))\\
&= -\Delta_H\varphi_i + \epsilon_{ikl}K^{jk}L_j{}^l + d^*_{ij}D^j\Omega + D_i(\mathring{\delta}(\boldsymbol{\varphi}) - \delta_{\boldsymbol{h}}(\boldsymbol{\varphi})), && (7.1.7)
\end{aligned}$$

which has an elliptic linearisation (with principal part $\mathring{\Delta}_H$). Just as we have to show vanishing of the De Turck term, we must of course verify, having constructed a solution φ_i of $\widetilde{X}_i = 0$, that $\mathring{\delta}(\boldsymbol{\varphi}) = 0$ so that the reduced equation is indeed equivalent to the CCE equation $X_i = 0$ —see the discussion below. Accordingly, we define the additional zero quantity

$$X \equiv \mathring{\delta}(\boldsymbol{\varphi}). \tag{7.1.8}$$

The original zero quantity, X_{ij} is then related to $\tilde{X}_i, X$ as follows

$$\tilde{X}_i - (dX)_i = -\tfrac{1}{2}\epsilon_{ijk}X^{jk},$$

or equivalently,

$$X_{ij} = -\tfrac{1}{2}\epsilon_{ij}{}^k(\tilde{X}_k - \tfrac{1}{2}(dX)_k). \tag{7.1.9}$$

To show that $X_{ij} = 0$ is satisfied, we will have to show that $\tilde{X}_i = 0$ and $X = 0$.

Remark 49. In defining $\widetilde{X}_i$, we could of course have chosen to instead a $D_i\delta_{\boldsymbol{h}}(\boldsymbol{\varphi})$ term, rather than the $D_i\mathring{\delta}(\boldsymbol{\varphi})$ term. If we had done so, the principal part would be precisely Δ_H. However, the approach taken above will prove convenient from the point of view of the sufficiency argument since we will be able to restrict X to the Banach space $\bar{H}^s$, defined with respect to the *fixed* background metric, $\mathring{\boldsymbol{h}}$. At the level of the linearised equations the two approaches are indistinguishable since the bracketed term in (7.1.7) will trivialise upon linearisation (recall that $\mathring{\varphi}_i = 0$ in the present case).

In accordance with the above discussion, we define the *auxiliary CCE map* as follows

$$\widetilde{\Xi}(\Omega, \sigma, s, \boldsymbol{\varphi}, \boldsymbol{\theta}, \boldsymbol{\chi}, \boldsymbol{u}^*, \boldsymbol{u}, \boldsymbol{h};\ \varphi, \theta, \phi, \boldsymbol{\psi}^*, \boldsymbol{\psi}) = \begin{pmatrix} \mathrm{tr}_{\mathring{\boldsymbol{h}}}\boldsymbol{P} \\ \mathring{\delta}(\boldsymbol{Z}) \\ \mathring{\delta}(\boldsymbol{W}) \\ \widetilde{X}_i \\ \mathring{\mathcal{D}}^*(\boldsymbol{X})_{ij} \\ \mathring{\mathcal{D}}^*(\boldsymbol{Y})_{ij} \\ \Lambda^*_i \\ \Lambda_i \\ V_{ij} - \tfrac{1}{2}(\mathcal{L}_{\boldsymbol{Q}}\boldsymbol{h})_{ij} \end{pmatrix},$$

where it is understood that we are substituting the ansatz

$$K_{ij} = K(\phi, \boldsymbol{\chi})_{ij} \equiv \chi_{ij} + \tfrac{1}{3}(\phi + \mathring{K})\mathring{h}_{ij}, \tag{7.1.10a}$$

$$L_{ij} = L(\theta, \boldsymbol{\theta})_{ij} \equiv \tfrac{\lambda}{6} h_{ij} + \theta_{ij} + \tfrac{1}{3}\theta\mathring{h}_{ij}, \tag{7.1.10b}$$

$$L_i = L(\varphi, \boldsymbol{\varphi})_i \equiv (d\varphi)_i + \mathrm{curl}(\boldsymbol{\varphi})_i, \tag{7.1.10c}$$

$$d_{ij} = d(\boldsymbol{u}(u), \boldsymbol{\psi})_{ij} \equiv \Pi_{\boldsymbol{h}}(\mathring{L}(\boldsymbol{u}) + \boldsymbol{\psi})_{ij}, \tag{7.1.10d}$$

$$d^*_{ij} = d^*(\boldsymbol{u}^*(u), \boldsymbol{\psi}^*)_{ij} \equiv \Pi_{\boldsymbol{h}}(\mathring{L}(\boldsymbol{u}^*) + \boldsymbol{\psi}^*)_{ij}, \tag{7.1.10e}$$

from the previous section. Recall that $\chi_{ij},\ \theta_{ij} \in \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})$ and that $\psi_{ij},\ \psi^*_{ij} \in \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})$. Here, Q_i is again a shorthand for the standard De Turck covector, given by $Q(\boldsymbol{h}; \mathring{\boldsymbol{h}})_i \equiv h_{ik}h^{jl}(\Gamma[\boldsymbol{h}]^k_{jl} - \mathring{\Gamma}^k_{jl})$, as used in Chapter 4. Note that the generalisations of the De Turck trick considered in Chapters 5 and 6 are also applicable in the case of the full CCEs, though we will not need them for the application to hyperbolic initial data sets considered later in this chapter. $\widetilde{\Xi} = 0$ is to be thought of as a system of equations for the determined fields

$$v = (\Omega,\ \sigma,\ s,\ \varphi_i,\ \theta_{ij},\ \chi_{ij},\ u^*_i,\ u_i,\ h_{ij}), \tag{7.1.11}$$

for a given choice of free data

$$u = (\varphi,\ \theta,\ \phi,\ \psi^*_{ij},\ \psi_{ij}).$$

It will be convenient to also define the map

$$w : (u, v) \mapsto w(u, v) \equiv \big(\Omega,\ \sigma,\ s,\ L(\varphi, \boldsymbol{\varphi})_i,\ L(\theta, \boldsymbol{\theta})_{ij}, K(\phi, \boldsymbol{\chi})_{ij},\ d^*(\boldsymbol{u}^*, \boldsymbol{\psi}^*)_{ij},\ d(\boldsymbol{u}, \boldsymbol{\psi})_{ij},\ h_{ij}\big)$$

which simply describes the substitution of u, v into the ansatz.

Remark 50. In the auxiliary map $\widetilde{\Xi}$, we are choosing to trace the zero quantity P_{ij} with respect to the background metric, $\mathring{\boldsymbol{h}}$, rather than the determined metric, $\boldsymbol{h}$. The reason for doing so is that in the sufficiency argument it will be convenient to work with the fixed bundle $\mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})$ (of which P_{ij} will be a section) rather than $\mathscr{S}^2_0(\mathcal{S}; \boldsymbol{h})$ which depends on the candidate solution.

Note that if $X_{ij} = 0$ and $X \equiv \mathring{\delta}(\boldsymbol{\varphi}) = 0$, then $\widetilde{X}_i = 0$. Conversely, if $\widetilde{X}_i = 0$ and $X \equiv \delta(\boldsymbol{\varphi}) = 0$ then $X_{ij} = 0$. However, we will only have access to $\widetilde{X}_i = 0$ via the auxiliary system, so then in order to conclude that $X_{ij} = 0$ we will have to show that $X = 0$ follows automatically from the auxiliary system (and the integrability relations). We will see that the argument is in the same spirit as that for the De Turck-reduced Gauss–Codazzi equation given in previous chapters.

The auxiliary map, $\widetilde{\Xi}$, is a second-order operator on the determined fields; the linearisation in the direction of the determined fields, $D_v\widetilde{\Xi}$, has principal part equivalent to that of[4]

$$\begin{pmatrix} \mathring{\Delta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{O}_1 & \mathring{\Delta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{O}_2 & 0 & \mathring{\Delta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathring{\Delta}_H & 0 & 0 & 0 & 0 & 0 \\ \mathcal{O}_3 & 0 & 0 & \mathcal{O}_4 & \mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}} & 0 & \mathcal{O}_5 & \mathcal{O}_6 & 0 \\ 0 & 0 & 0 & \mathcal{O}_7 & 0 & \mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}} & \mathcal{O}_8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathring{\delta} \circ \mathring{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathring{\delta} \circ \mathring{L} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\mathring{\Delta}_L \end{pmatrix} \begin{pmatrix} \breve{\Omega} \\ \breve{\sigma} \\ \breve{s} \\ \breve{\varphi}_i \\ \breve{\theta}_{ij} \\ \breve{\chi}_{ij} \\ \breve{u}^*_i \\ \breve{u}_i \\ \gamma_{ij} \end{pmatrix},$$

where the operators $\mathcal{O}_I$ are certain linear second order operators whose explicit form is not needed here due to the semi-decoupled nature of the principal part. Indeed, to verify ellipticity, we begin with the first, fourth, seventh and eighth rows of the corresponding principal symbol map; we find that for an element of the kernel, we have $\breve{\Omega} = \breve{\varphi}_i = \breve{u}^*_i = \breve{u}_i = 0$ (since $\mathring{\Delta}$, $\mathring{\Delta}_H$ and $\mathring{\delta} \circ \mathring{L}$ are elliptic) and the contributions from the $\mathcal{O}_I$ operators then trivialise, leaving only the diagonal entries which are elliptic by construction (i.e. by choice of ansatz and gauge reduction). $D_v\widetilde{\Xi}$ is therefore a second order elliptic operator and is Fredholm on closed $\mathcal{S}$ (as considered here).

Having solved the auxiliary system to obtain a *candidate solution*, one must still verify that it indeed satisfies the CCEs by showing that

$$P_{ij} = 0, \qquad X_{ijk} = Y_{ijk} = 0, \qquad X = 0, \qquad Q_i = 0.$$

As mentioned before, this will essentially be guaranteed by the integrability conditions in a way that is analogous to the argument for the ECEs, given in Chapter 4.

Remark 51. Note that, when constructing the auxiliary map, we could also use the CCEs to substitute for some of the higher-derivative terms —e.g. those second-order terms appearing in the operators $\mathcal{O}_I$— which would lead to simplified principal part. This is effected by adding additional linear expressions of the CCE quantities to the map $\widetilde{\Xi}$, given above. It is not clear to what extent this is useful for the subsequent analysis. The above auxiliary map is sufficient for our purposes, here.

7.2 Application to conformally rigid hyperbolic background initial data sets

In this section, we return to the conformally rigid hyperbolic background initial data sets considered in Chapter 4, thought of now as solutions of the full CCEs by fixing the following background fields

$$\mathring{\Omega} = 1, \quad \mathring{\sigma} = 0, \quad \mathring{s} = \frac{\mathring{\lambda}}{6}, \quad \mathring{L}_i = 0, \quad \mathring{L}_{ij} = \frac{\lambda}{6}\mathring{h}_{ij}, \quad \mathring{d}_{ij} = 0, \quad \mathring{d}^*_{ij} = 0 \tag{7.2.1}$$

—see Proposition 1. We will call such a solution to the full CCEs conformally rigid, hyperbolic, as before. The background solution can be realised (uniquely) in the form of the given ansatz —i.e.

[4] Of course, the operators here contain lower-order terms (e.g. curvature terms) which do not enter into the principal part, strictly speaking.

$w(\mathring{u}, \mathring{v})$ with

$$\mathring{u} = (0, 0, 0, \mathbf{0}, \mathbf{0}), \qquad \mathring{v} = (1, 0, \tfrac{\mathring{\lambda}}{6}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathring{\boldsymbol{h}}).$$

Recall that the Hamiltonian constraint fixes the cosmological constant to $\mathring{\lambda} = \frac{1}{3}(\mathring{K}^2 - 9)$. In this section we show the existence of non-linear perturbative solutions of the full CCEs around these background solutions, using the extension of the Friedrich–Butscher method outlined in the previous section. This section should be thought of as a proof-of-concept of the proposed solution scheme.

More precisely, the main result is as follows (the Banach spaces $\mathcal{X}^k, \mathcal{Y}^k$ will be fixed in the forthcoming section):

Theorem 7. Consider a closed conformally rigid solution of the CCEs of the form (7.2.1), with constant mean extrinsic curvature $\mathring{K}$ satisfying

$$\beta \notin \mathit{Spec}\big(- \mathring{\Delta} : C^\infty(\mathcal{S}) \to C^\infty(\mathcal{S})\big). \tag{7.2.2}$$

and cosmological constant $\mathring{\lambda} = \frac{1}{3}(\mathring{K}^2 - 9)$. Then, there exists an open neighbourhood $\mathcal{U} \subset \mathcal{X}^k$ of $\mathring{u} = 0$, an open neighbourhood $\mathcal{W} \subset \mathcal{Y}^k$ of $\mathring{v}$ and a smooth map $\nu : \mathcal{U} \to \mathcal{W}$ such that, defining

$$u \equiv (\varphi, \theta, \phi, \boldsymbol{\psi}^*, \boldsymbol{\psi}), \qquad \nu(u) \equiv \big(\Omega(u), s(u), \sigma(u), \boldsymbol{\varphi}(u), \boldsymbol{\chi}(u), \boldsymbol{u}^*(u), \boldsymbol{u}(u), \boldsymbol{h}(u)\big),$$

the following assertions hold:

(i) for each $u \in \mathcal{U}$,

$$\begin{aligned} w(u) \equiv w(u, \nu(u)) = \big(&\Omega(u), s(u), \sigma(u), L(\varphi, \boldsymbol{\varphi}(u))_i, L(\theta, \boldsymbol{\theta}(u))_{ij}, \\ &K(\phi, \boldsymbol{\chi}(u))_{ij}, d^*(\boldsymbol{u}^*(u), \boldsymbol{\psi}^*)_{ij},\ d(\boldsymbol{u}(u), \boldsymbol{\psi})_{ij},\ h(u)_{ij}\big), \end{aligned}$$

is a solution to the Conformal Constraint Equations with cosmological constant

$$\lambda = 6\Omega s + 3\sigma^2 - 3\|d\Omega\|^2_{\boldsymbol{h}}$$

which has the same sign as $\mathring{\lambda}$. In particular, $\mathring{\delta}(\boldsymbol{\varphi}) = 0$;

(ii) the map $u \mapsto w(u)$ is injective for $\mathring{K} \neq 0$. Moreover, it is injective for $\mathring{K} = 0$ if we restrict the free datum ϕ to the sub-Banach space $\bar{H}^{k-1}(\mathscr{C}(\mathcal{S}))$.

(iii) each solution $w(u)$ of the CCEs results in a solution of the ECEs of the form considered in Theorem 2, with free data $(\tilde{\phi},\ \bar{T}_{ij},\ T_{ij})$ given by

$$\begin{aligned} \tilde{\phi} &= \Omega\phi - \sigma(\mathrm{tr}_{\mathring{\boldsymbol{h}}}\boldsymbol{h}), \\ \bar{T}_{ij} &= -\mathring{L}(\bar{\boldsymbol{X}})_{ij} + \Omega\mathring{L}(\boldsymbol{u}^*) + \Omega\psi^*_{ij}, \\ T_{ij} &= -\mathring{L}(\boldsymbol{X})_{ij} + \Omega\mathring{L}(\boldsymbol{u}) + \Omega\psi_{ij}, \end{aligned}$$

with $\bar{X}_i,\ X_i$ given implicitly by the equations

$$\begin{aligned} \mathring{\delta} \circ \mathring{L}(\bar{\boldsymbol{X}})_i &= \mathring{\delta}(\Omega\mathring{L}(\boldsymbol{u}^*))_i + (d\Omega)^j\psi^*_{ij}, \\ \mathring{\delta} \circ \mathring{L}(\boldsymbol{X})_i &= \mathring{\delta}(\Omega\mathring{L}(\boldsymbol{u}))_i + (d\Omega)^j\psi_{ij}. \end{aligned}$$

Conclusions (i) and (ii) will be proved in Propositions 20 and 23 of Sections 7.2.1 and 7.2.2. Conclusion (iii) will be addressed in Section 7.2.3.

Note that the construction of TT tensors described in Section 4.4 can also be used to give a parametrisation of (smooth) free data $\psi_{ij},\ \psi^*_{ij} \in \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})$.

7.2.1 The existence of candidate solutions

Note that in terms of the perturbed zero-quantities[5], $\breve{W}_i,\ \breve{Z}_i$, the linearisations of the auxiliary equations $\mathring{\delta}(\boldsymbol{W}) = \mathring{\delta}(\boldsymbol{Z}) = 0$ (in the direction of the determined fields) are given simply by $\mathring{\delta}(\breve{\boldsymbol{W}}) = \mathring{\delta}(\breve{\boldsymbol{Z}}) = 0$, and therefore their image sits within the space of mean-zero functions —i.e. in a space of the form $\bar{H}^s$. It is reasonable therefore to also restrict the fields σ, s to the space $\bar{H}^k$ (for some suitably large $k \in \mathbb{N}$). In the following, we will use the following convenient short-hands:

$$\breve{L}_i = \mathring{\epsilon}_i{}^{jk}\mathring{D}_j\breve{\varphi}_k, \qquad \breve{d}_{ij} = \mathring{L}(\breve{\boldsymbol{u}})_{ij}, \qquad \breve{d}^*_{ij} = \mathring{L}(\breve{\boldsymbol{u}}^*)_{ij}$$

representing the perturbation of the corresponding fields in the direction of the determined component of their respective decompositions —see previous section. Note in particular that $\mathring{\delta}(\breve{\boldsymbol{L}}) = 0$, which we use below in computing $\mathring{\delta}(\breve{\boldsymbol{Z}})$.

The linearisation of the auxiliary system in the direction of the determined fields, denoted D_v, is given by

$$D_v\widetilde{\Xi}\cdot(\breve{\Omega}, \breve{\sigma}, \breve{s}, \breve{\varphi}_i, \breve{\theta}_{ij}, \breve{\chi}_{ij}, \breve{u}^*_i, \breve{u}_i, \gamma_{ij})$$
$$= \begin{pmatrix} \breve{P}_k{}^k \\ \mathring{\delta}(\breve{\boldsymbol{Z}}) \\ \mathring{\delta}(\breve{\boldsymbol{W}}) \\ \breve{X}_i \\ \mathring{\mathcal{D}}^*(\breve{\boldsymbol{X}})_{jk} \\ \mathring{\mathcal{D}}^*(\breve{\boldsymbol{Y}})_{jk} \\ \breve{\Lambda}^*_i \\ \breve{\Lambda}_i \\ \breve{V}_{ij} \end{pmatrix} \equiv \begin{pmatrix} \mathring{\Delta}\breve{\Omega} + \frac{1}{2}\lambda\breve{\Omega} - 3\breve{s} - \mathring{K}\breve{\sigma} \\ \mathring{\Delta}\breve{\sigma} - \frac{1}{3}\mathring{K}\mathring{\Delta}\breve{\Omega} \\ \mathring{\Delta}\breve{s} + \frac{\lambda}{6}\mathring{\Delta}\breve{\Omega} \\ -\mathring{\Delta}_H\breve{\varphi}_i \\ \mathring{\mathcal{D}}^*\circ\mathring{\mathcal{D}}(\breve{\boldsymbol{\theta}})_{ij} + \frac{1}{6}\mathring{K}\mathring{L}(\breve{L})_{ij} \\ \mathring{\mathcal{D}}^*\circ\mathring{\mathcal{D}}(\breve{\boldsymbol{\chi}} - \frac{1}{3}\mathring{K}\bar{\boldsymbol{\gamma}})_{jk} + \frac{1}{2}\mathring{L}(\breve{\boldsymbol{L}})_{jk} + \mathring{\epsilon}_{il(j}\mathring{D}^l\mathring{L}(\breve{\boldsymbol{u}}^*)_{k)}{}^i \\ \mathring{\delta}\circ\mathring{L}(\breve{\boldsymbol{u}}^*)_i \\ \mathring{\delta}\circ\mathring{L}(\breve{\boldsymbol{u}})_i \\ \frac{1}{2}\mathring{\Delta}_L\gamma_{ij} - \mathring{L}(\breve{\boldsymbol{u}})_{ij} - \frac{1}{9}\left(6\lambda - \mathring{K}^2\right)\gamma_{ij} \\ -\left(-1 + \frac{1}{9}\mathring{K}^2\right)\gamma^k{}_k\mathring{h}_{ij} + \frac{1}{3}\mathring{K}\breve{\chi}_{ij} - \breve{\theta}_{ij} \end{pmatrix}$$

As with $\breve{W}_i,\ \breve{Z}_i$ above, the breved zero quantities $\breve{P}_{ij}, \breve{Z}_i, \cdots$ denote the linearisations of the zero quantities in the direction of the determined fields. The full expressions will not be needed here. When studying the kernel of the above map, it is convenient to substitute for the $\mathring{\Delta}\breve{\Omega}$ terms to obtain

$$\mathring{\Delta}\breve{\Omega} + \tfrac{1}{2}\lambda\breve{\Omega} - 3\breve{s} - \mathring{K}\breve{\sigma} = 0, \tag{7.2.3a}$$
$$\mathring{\Delta}\breve{\sigma} + \tfrac{1}{6}\mathring{K}\lambda\breve{\Omega} - \mathring{K}\breve{s} - \tfrac{1}{3}\mathring{K}^2\breve{\sigma} = 0, \tag{7.2.3b}$$
$$\mathring{\Delta}\breve{s} - \tfrac{1}{12}\lambda^2\breve{\Omega} + \tfrac{1}{2}\lambda\breve{s} + \tfrac{1}{6}\mathring{K}\lambda = 0\breve{\sigma}. \tag{7.2.3c}$$

Rather than calculating the L^2-adjoint of $D_v\widetilde{\Xi}$, we choose to substitute the operators defined by (7.2.3a)–(7.2.3c) and calculate the adjoint of the resulting map. The vanishing of the resulting

[5] That is to say, the linearsiation of the zero quantities in the direction of the determined fields.

adjoint map is given by

$$\mathring{\Delta}\Omega' + \tfrac{1}{2}\lambda\Omega' - \tfrac{1}{12}\lambda^2 s' + \tfrac{1}{6}\mathring{K}\lambda\sigma' = 0, \tag{7.2.4a}$$
$$\mathring{\Delta}\sigma' - \mathring{K}\Omega' + \tfrac{1}{6}\mathring{K}\lambda s' - \tfrac{1}{3}\mathring{K}^2\sigma' = 0, \tag{7.2.4b}$$
$$\mathring{\Delta}s' - 3\Omega' + \tfrac{1}{2}\lambda s' - \mathring{K}\sigma' = 0, \tag{7.2.4c}$$
$$-\mathring{\Delta}_H\varphi'_k - \mathring{\epsilon}_{kjl}\mathring{D}^l\mathring{D}_i\chi'^{ij} - \tfrac{1}{3}\mathring{K}\mathring{\epsilon}_{kjl}\mathring{D}^l\mathring{D}_i\theta'^{ij} = 0, \tag{7.2.4d}$$
$$\mathring{\mathcal{D}}^*\circ\mathring{\mathcal{D}}(\boldsymbol{\theta}')_{ij} - \gamma'_{ij} + \tfrac{1}{3}\gamma'_k{}^k\mathring{h}_{ij} = 0, \tag{7.2.4e}$$
$$\mathring{\mathcal{D}}^*\circ\mathring{\mathcal{D}}(\boldsymbol{\chi}')_{ij} + \tfrac{1}{3}\mathring{K}\gamma'_{ij} - \tfrac{1}{9}\mathring{K}\mathring{h}_{ij}\gamma'^k{}_k = 0, \tag{7.2.4f}$$
$$\mathring{\delta}\circ\mathring{L}(v')_i - \mathring{\epsilon}_{ikl}\mathring{D}_j\mathring{D}^l\chi'^{jk} = 0, \tag{7.2.4g}$$
$$\mathring{\delta}\circ\mathring{L}(u')_i - \tfrac{2}{3}\mathring{D}_i\gamma'^j{}_j + 2\mathring{D}_j\gamma'^j_i = 0, \tag{7.2.4h}$$
$$\tfrac{1}{2}\mathring{\Delta}_L\gamma'_{ij} - \tfrac{1}{9}\left(6\lambda - \mathring{K}^2\right)\gamma'_{ij} - \left(-1 + \tfrac{1}{9}\mathring{K}^2\right)\gamma'_k{}^k\mathring{h}_{ij} - \tfrac{1}{3}\mathring{K}\left(-\mathring{\Delta}\breve{\chi}'_{ij} + \mathring{D}_k\mathring{D}_{(i}\breve{\chi}_{j)}{}^k\right) = 0. \tag{7.2.4i}$$

Clearly the above system will be equivalent to $(D_v\widetilde{\Xi})^* = 0$. As mentioned above, since the linearised auxiliary equations for $\breve{s}$, $\breve{\sigma}$ are total (background) divergences, they map into the space of mean-zero functions. Accordingly, we will take $s', \sigma' \in \bar{H}^l$ (for the appropriate $l \in \mathbb{N}$). We define, for $k \geq 4$, the following Banach spaces

$$\mathcal{X}^k \equiv \bar{H}^{k-1}(\mathscr{C}(\mathcal{S})) \times H^{k-1}(\mathscr{C}(\mathcal{S})) \times H^{k-1}(\mathscr{C}(\mathcal{S})) \times H^{k-1}(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}})) \times H^{k-1}(\mathscr{S}_{TT}(\mathcal{S};\mathring{\boldsymbol{h}})),$$

$$\begin{aligned}\mathcal{Y}^k \equiv\ & H^k(\mathscr{C}(\mathcal{S})) \times \bar{H}^k(\mathscr{C}(\mathcal{S})) \times \bar{H}^k(\mathscr{C}(\mathcal{S})) \times H^k(\Lambda^1(\mathcal{S})) \\ & \times H^k(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})) \times H^k(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})) \times H^k(\Lambda^1(\mathcal{S})) \times H^k(\Lambda^1(\mathcal{S})) \times H^k(\mathscr{S}^2(\mathcal{S})),\end{aligned}$$

$$\begin{aligned}\mathcal{Z}^k \equiv\ & H^{k-2}(\mathscr{C}(\mathcal{S})) \times \bar{H}^{k-2}(\mathscr{C}(\mathcal{S})) \times \bar{H}^{k-2}(\mathscr{C}(\mathcal{S})) \times H^{k-2}(\Lambda^1(\mathcal{S})) \\ & \times H^{k-2}(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})) \times H^{k-2}(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})) \times H^{k-2}(\Lambda^1(\mathcal{S})) \times H^{k-2}(\Lambda^1(\mathcal{S})) \times H^{k-2}(\mathscr{S}^2(\mathcal{S})),\end{aligned}$$

where the ordering is such that

$$(\varphi,\ \theta,\ \phi,\ \psi^*_{ij},\ \psi_{ij}) \in \mathcal{X}^k,$$

and

$$(\Omega,\ s,\ \sigma,\ \varphi_i,\ \chi_{ij},\ \theta_{ij},\ u_i,\ u^*_i,\ h_{ij}) \in \mathcal{Y}^k.$$

Again, the spaces $\mathcal{X}^k$, $\mathcal{Y}^k$, $\mathcal{X}^k$ are equipped with the obvious norms —given by summation of the norms of each Banach space in the product— and denoted by $\|\cdot\|_{\mathcal{X}^k}, \|\cdot\|_{\mathcal{Y}^k}, \|\cdot\|_{\mathcal{Z}^k}$.

Remark 52. Note that $D_v\widetilde{\Xi}$ maps $\mathcal{Y}^k$ into $\mathcal{Z}^k$, and that $\widetilde{\Xi}$ maps $\mathcal{X}^k \times \mathcal{Y}^k$ into $\mathcal{Z}^k$, which can be easily checked using the Schauder ring property —see Section 2.3.2.

Proposition 20. For a conformally rigid hyperbolic background solution, the linearised auxiliary CCE map is an isomorphism of Banach spaces $\mathcal{Y}^k$, $\mathcal{Z}^k$ for $k \geq 4$. Hence, the IFT guarantees the existence of open subsets $\mathcal{U} \subset \mathcal{X}^k$, $\mathcal{V} \subset \mathcal{Y}^k$ and a map $\nu : \mathcal{U} \to \mathcal{V}$ mapping free data to solutions of the auxiliary equations —i.e. such that $\widetilde{\Xi}(\nu(u); u) = 0$ for all $u \in \mathcal{U}$.

Proof.

Proof of injectivity: First note that $\breve{\Lambda}_i = \breve{\Lambda}^*_i = 0$ imply $\breve{u}_i = \breve{u}^*_i = 0$, since $\mathring{\boldsymbol{h}}$ admits no conformal Killing vectors. Also, $\breve{\bar{X}}_i = 0$ implies $\breve{\varphi}_i = 0$, since $(\mathcal{S}, \mathring{\boldsymbol{h}})$ admits no harmonic 1−forms. Substituting into $\mathring{\mathcal{D}}^*(\boldsymbol{X})_{ij} = \mathring{\mathcal{D}}^*(\boldsymbol{Y})_{ij} = 0$ one finds immediately that

$$\breve{\theta}_{ij} = 0, \qquad \breve{\chi}_{ij} = \tfrac{1}{3}\mathring{K}\bar{\gamma}_{ij},$$

since $\mathring{\boldsymbol{h}}$ admits no nontrivial tracefree Codazzi tensors. Substituting into $\breve{\mathring{V}}_{ij} = 0$, we obtain

$$0 = \tfrac{1}{2}\mathring{\Delta}_L\gamma_{ij} - \tfrac{1}{9}\left(6\lambda - \mathring{K}^2\right)\gamma_{ij} - \left(-1 + \tfrac{1}{9}\mathring{K}^2\right)\gamma_k{}^k\mathring{h}_{ij} - \tfrac{1}{9}\mathring{K}^2\bar{\gamma}_{ij},$$

which decomposes into the following trace and tracefree parts

$$\left(\mathring{\Delta} + \beta\right)\gamma = 0, \tag{7.2.5a}$$

$$\left(\mathring{\Delta}_L + 4\right)\bar{\gamma}_{ij} = 0, \tag{7.2.5b}$$

where $\beta = -4 + (8/9)\mathring{K}^2$. By assumption $\beta \notin \operatorname{spec}(-\mathring{\Delta})$, and so we see that $\gamma = 0$. Recall from Proposition 5 of Chapter 4 that $\mathring{P}_L \equiv \mathring{\Delta}_L + 4 : H^k(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})) \to H^{k-2}(\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}}))$ is an isomorphism. Hence we find that $\bar{\gamma}_{ij} = 0$. It remains to analyse (7.2.3a)–(7.2.3c). First, note that by integrating (7.2.3c), we immediately have that $\lambda\breve{\Omega}$ integrates to zero —i.e. $\lambda\breve{\Omega} \in \bar{H}^k(\mathscr{C}(\mathcal{S}))$. Then, it follows from (7.2.3a) and (7.2.3c) that

$$\mathring{\Delta}\left(\lambda\breve{\Omega} + 6\breve{s}\right) = 0,$$

implying $\lambda\breve{\Omega} + 6\breve{s}$ is constant and hence, since it is an element of $H^k(\mathscr{C}(\mathcal{S}))$, we have $\lambda\breve{\Omega} + 6\breve{s} = 0$. Substituting into (7.2.3a) and (7.2.3b),

$$0 = \mathring{\Delta}\breve{\Omega} + \lambda\breve{\Omega} - \mathring{K}\breve{\sigma}, \tag{7.2.6a}$$

$$0 = \mathring{\Delta}\breve{\sigma} + \tfrac{1}{3}\mathring{K}\lambda\breve{\Omega} - \tfrac{1}{3}\mathring{K}^2\breve{\sigma}. \tag{7.2.6b}$$

Taking linear combinations, we obtain

$$\mathring{\Delta}\left(\breve{\sigma} - \tfrac{1}{3}\mathring{K}\breve{\Omega}\right) = 0,$$

from which we see $\breve{\sigma} - \frac{1}{3}\mathring{K}\breve{\Omega}$ is constant hence zero; it follows that $\breve{\sigma} = \frac{1}{3}\mathring{K}\breve{\Omega}$. Substituting into (7.2.6b),

$$0 = \mathring{\Delta}\breve{\sigma} + \left(\lambda - \tfrac{1}{3}\mathring{K}^2\right)\breve{\sigma} = (\mathring{\Delta} - 3)\breve{\sigma},$$

and so we have $\breve{\sigma} = 0$, from which it follows that $\breve{\Omega} = \breve{s} = 0$. Hence, the linearised auxiliary CCE map is injective.

Proof of surjectivity: Since the linearised auxiliary constraint map is (by construction) second-order elliptic, it suffices by the Fredholm alternative to show that the adjoint is injective. Integrating (7.2.4c), we immediately find that Ω' integrates to zero —i.e. $\Omega' \in \bar{H}^{k-2}(\mathscr{C}(\mathcal{S}))$. Then, combining (7.2.4a) and (7.2.4c), we find that

$$\mathring{\Delta}\left(6\Omega' + \lambda s'\right) = 0,$$

so $6\Omega' + \lambda s'$ is constant and hence, since it is an element of $\bar{H}^{k-2}(\mathscr{C}(\mathcal{S}))$, it is zero. Substituting for Ω' in (7.2.4a) and (7.2.4b), we obtain

$$\mathring{\Delta}s' + \lambda s' - \mathring{K}\sigma' = 0, \tag{7.2.7a}$$

$$\mathring{\Delta}\sigma' + \tfrac{1}{3}\mathring{K}\lambda s' - \tfrac{1}{3}\mathring{K}^2\sigma' = 0. \tag{7.2.7b}$$

Note that, substituting into (7.2.4c), one obtains equation (7.2.7a) once more. Moreover, we find that

$$\mathring{\Delta}\left(\sigma' - \tfrac{1}{3}\mathring{K}s'\right) = 0,$$

so that $\sigma' - \frac{1}{3}\mathring{K}s' = 0$, whence (7.2.7a) and (7.2.7b) both reduce to

$$\mathring{\Delta}s' - 3s' = 0$$

from which it follows by positive-definiteness of $(-\mathring{\Delta} + 3)$ that $s' = 0$, and hence from the above that $\Omega' = s' = \sigma' = 0$. Substituting into (7.2.4d),

$$\mathring{\Delta}_H \varphi'_i = 0,$$

and hence we see that $\varphi'_i = 0$, again since $(\mathcal{S}, \mathring{\boldsymbol{h}})$ admits no harmonic 1−forms. Note that

$$\mathring{\mathcal{D}}^* \circ \mathring{\mathcal{D}}(\boldsymbol{\eta})_{ij} \equiv -\mathring{\Delta}\eta_{ij} + \mathring{D}_k \mathring{D}_{(i}\boldsymbol{\eta}_{j)}{}^k - \tfrac{1}{3}\mathring{h}_{ij}\mathring{D}^k\mathring{D}^l\boldsymbol{\eta}_{ij}$$

for any $\boldsymbol{\eta} \in \mathscr{S}^2(\mathcal{S})$. Hence, we can rearrange (7.2.4f) and substitute into (7.2.4i) to obtain

$$0 = \tfrac{1}{2}\mathring{\Delta}_L\gamma'_{ij} + 2\gamma'_{ij} - \left(-1 + \tfrac{4}{27}\mathring{K}^2\right)\gamma'_k{}^k\mathring{h}_{ij} - \tfrac{1}{9}\mathring{K}\mathring{h}_{ij}\mathring{D}^k\mathring{D}^l\breve{\chi}_{kl},$$

which decomposes to give the following trace and tracefree parts (with respect to $\mathring{\boldsymbol{h}}$):

$$\left(\mathring{\Delta}_L + 4\right)\bar{\gamma}'_{ij} = 0, \tag{7.2.8a}$$

$$\left(\mathring{\Delta} + \beta\right)\gamma' + \tfrac{2}{3}\mathring{K}\mathring{D}^k\mathring{D}^l\breve{\chi}_{kl} = 0, \tag{7.2.8b}$$

where γ', $\bar{\gamma}'_{ij}$ denote the trace and tracefree parts of γ'_{ij}, respectively. As shown previously, $\mathring{P}_L \equiv \mathring{\Delta}_L + 4 : H^{k-2}(\mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})) \to H^{k-4}(\mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}}))$ is injective for $k \geq 4$, so it follows immediately that $\bar{\gamma}'_{ij} = 0$. Substituting into (7.2.4f) and (7.2.4e) and using the fact that $\mathring{\boldsymbol{h}}$ admits no nontrivial tracefree Codazzi tensors, we see that $\breve{\chi}_{ij} = \breve{\theta}_{ij} = 0$. Substituting into (7.2.8b), and using the fact that, by assumption $\beta \notin \mathrm{spec}(-\mathring{\Delta})$, we find that $\gamma' = 0$.

In light of the above, equations (7.2.4g)–(7.2.4h) reduce to

$$\mathring{\delta} \circ \mathring{L}(\boldsymbol{u}')_{ij} = \mathring{\delta} \circ \mathring{L}(\boldsymbol{v}')_{ij} = 0.$$

Hence, $u'_i = v'_i = 0$, since $\mathring{\boldsymbol{h}}$ admits no conformal Killing vectors. Hence, the adjoint is injective and therefore, by the Fredholm alternative, the auxiliary map is surjective. □

Proposition 21. The linearisation of the auxiliary CCE map in the direction of the free data, $D_u\widetilde{\Xi} : \mathcal{X}^k \to \mathcal{Z}^k$ is

(i) injective (for $k \geq 4$) on $\mathcal{X}^k$, provided $\mathring{K} \neq 0$;

(ii) injective (for $k \geq 4$) for $\mathring{K} = 0$, provided ϕ is restricted to $\bar{H}^{k-1}(\mathscr{C}(\mathcal{S}))$ —i.e. if $\mathcal{X}^k$ is modified to

$$\mathcal{X}^k \equiv \bar{H}^{k-2}(\mathscr{C}(\mathcal{S})) \times H^{k-1}(\mathscr{C}(\mathcal{S})) \times \bar{H}^{k-1}(\mathscr{C}(\mathcal{S})) \times H^{k-1}(\mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})) \times H^{k-1}(\mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})).$$

In either of the above cases, the IFT guarantees that the map from free data to solutions, $\nu : \mathcal{X}^k \to \mathcal{Z}^k$, is also injective.

Proof. Direct computation shows that

$$D_u\widetilde{\Xi}\cdot(\breve{\varphi},\breve{\theta},\breve{\phi},\breve{\boldsymbol{\psi}}^*,\breve{\boldsymbol{\psi}}) = \begin{pmatrix} \breve{\theta} \\ 0 \\ \mathring{\Delta}\breve{\varphi} \\ -\frac{1}{6}\mathring{L}(d(\breve{\theta}+\mathring{K}\breve{\varphi}))_{ij} \\ -\frac{1}{6}\mathring{L}(d(\breve{\phi}+3\breve{\varphi}))_{ij}+\mathring{\mathcal{R}}(\breve{\boldsymbol{\psi}}^*)_{ij} \\ \mathring{\delta}(\breve{\boldsymbol{\psi}})_i \\ \mathring{\delta}(\breve{\boldsymbol{\psi}}^*)_i \\ -\breve{\psi}_{ij}+\frac{4}{9}(\mathring{K}\breve{\phi}-3\breve{\theta})\mathring{h}_{ij} \end{pmatrix}.$$

Then we have immediately from the first and third line that $\breve{\theta}=0$ and $\breve{\varphi}=0$, since $\breve{\varphi}\in\bar{H}^{k-1}(\mathscr{C}(\mathcal{S}))$. At this point, the fourth line trivialises and we are left with

$$\begin{aligned}
&\mathring{\mathcal{R}}(\breve{\boldsymbol{\psi}}^*)_{ij}-\tfrac{1}{6}\mathring{L}(d\breve{\phi})_{ij}=0,\\
&\mathring{\delta}(\breve{\boldsymbol{\psi}})_i=0,\\
&\mathring{\delta}(\breve{\boldsymbol{\psi}}^*)_i=0,\\
&\breve{\psi}_{ij}-\tfrac{4}{9}\mathring{K}\breve{\phi}\mathring{h}_{ij}=0.
\end{aligned}$$

This is equivalent to the system arising in the ECE case. By the same argument then (see Theorem 2) we have that, in the case $\mathring{K}\neq 0$, $\breve{\phi}=0$, $\breve{\psi}_{ij}=\breve{\psi}^*_{ij}=0$, while in the case $\mathring{K}=0$, $\breve{\phi}=0$, $\breve{\psi}_{ij}=\breve{\psi}^*_{ij}=0$ provided we restrict ϕ to the space $\bar{H}^{k-1}(\mathscr{C}(\mathcal{S}))$. Hence, By the IFT, ν is also injective. □

7.2.2 The sufficiency argument

We suppose now we have constructed a candidate solution,

$$w(u)=(\Omega,\ \sigma,\ s,\ L_i,\ L_{ij},\ K_{ij},\ d^*_{ij},\ d_{ij},\ h_{ij})$$

—i.e. a tuple for which the corresponding zero quantities satisfy

$$\mathrm{tr}_{\mathring{\boldsymbol{h}}}\boldsymbol{P}=0,\qquad \mathring{\delta}(\boldsymbol{Z})=0,\qquad \mathring{\delta}(\boldsymbol{W})=0,\qquad \bar{X}_i=0,\qquad \mathring{\mathcal{D}}^*(\boldsymbol{X})_{ij}=0,\qquad \mathring{\mathcal{D}}^*(\boldsymbol{X})_{ij}=0,$$

$$\Lambda^*_i=0,\qquad \Lambda_i=0,\qquad V_{ij}=\tfrac{1}{2}\mathcal{L}_{\boldsymbol{Q}}h_{ij}=-\delta^*(\boldsymbol{Q})_{ij}.$$

Rearranging (7.1.9), we have that $\bar{X}_i=0$ is equivalent to

$$X_{ij}=\tfrac{1}{2}\epsilon_{ij}{}^k(dX)_k,$$

and, since $U_{ij}=V_{ij}-\frac{1}{4}(\mathrm{tr}_{\boldsymbol{h}}\boldsymbol{V})h_{ij}$, we have

$$U_{ij}=-\delta^*(\boldsymbol{Q})_{ij}-\tfrac{1}{2}\delta(\boldsymbol{Q})h_{ij}. \tag{7.2.9}$$

Substituting the above into the integrability relations (7.1.2f)–(7.1.2c), (7.1.2e)–(7.1.2f), one obtains

$$\epsilon_{lik}D^kP_j{}^i - \epsilon_{jli}W^i + \tfrac{1}{2}\epsilon_{lik}(\Omega X^{ik}{}_j - 2K_j{}^kZ^i - \sigma Y^{ik}{}_j) = D^i\Omega(\epsilon_{jlk}U_i{}^k + \epsilon_{lik}U_j{}^k), \tag{7.2.10a}$$

$$2D_{[i}Z_{j]} - 2K_{k[i}P_{j]}{}^k + (D^k\Omega)Y_{ijk} = \epsilon_{ijk}\Omega D^kX, \tag{7.2.10b}$$

$$2D_{[i}W_{j]} - 2L_{[i}Z_{j]} + 2L_{k[i}P_{j]}{}^k - (D^k\Omega)X_{ijk} = -\sigma\epsilon_{ijk}D^kX, \tag{7.2.10c}$$

$$\epsilon_{ijk}D^kX^{ij}{}_l + 2\epsilon_{ljk}d_i{}^kP^{ij} + 2d^*_{li}Z^i + \epsilon_{ijk}L^iY^{jk}{}_l = 2\epsilon_{ljk}L_i{}^kU^{ij} + 2K_{li}D^iX, \tag{7.2.10d}$$

$$\epsilon_{ijk}D^kY^{ij}{}_l = -2\epsilon_{ljk}K^{ij}U_i{}^k + 2D_lX. \tag{7.2.10e}$$

Equation (7.2.10a) is equivalent to

$$2D_{[i}P_{j]k} - \Omega X_{ijk} + \sigma Y_{ijk} - 2h_{k[i}W_{j]} - 2K_{k[i}Z_{j]} = -2D_{[i}\Omega U_{j]k} + 2(D_l\Omega)h_{k[i}U_{j]}{}^l. \tag{7.2.11}$$

Note that we should of course substitute for the U_{ij} terms using (7.2.9), but for ease of presentation we leave them as they are. In what follows, it will prove convenient to define the map

$$\mathcal{Q}_w : \begin{pmatrix} \mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}}) \\ \Lambda^1(\mathcal{S}) \\ \Lambda^1(\mathcal{S}) \\ \mathcal{J}(\mathcal{S}) \\ \mathcal{J}(\mathcal{S}) \end{pmatrix} \to \begin{pmatrix} \mathscr{C}(\mathcal{S}) \\ \mathscr{C}(\mathcal{S}) \\ \mathscr{S}^2_0(\mathcal{S};h) \\ \mathscr{S}^2_0(\mathcal{S};h) \end{pmatrix} \oplus \begin{pmatrix} \mathcal{J}(\mathcal{S}) \\ \Lambda^2(\mathcal{S}) \\ \Lambda^2(\mathcal{S}) \\ \Lambda^1(\mathcal{S}) \\ \Lambda^1(\mathcal{S}) \end{pmatrix}$$

given as follows

$$\mathcal{Q}_w \begin{pmatrix} P_{ij} \\ Z_i \\ W_i \\ X_{ijk} \\ Y_{ijk} \end{pmatrix} = \begin{pmatrix} \mathring{\delta}(\boldsymbol{Z}) \\ \mathring{\delta}(\boldsymbol{W}) \\ \mathring{\mathcal{D}}^*(\boldsymbol{X})_{ij} \\ \mathring{\mathcal{D}}^*(\boldsymbol{Y})_{ij} \end{pmatrix} \oplus \begin{pmatrix} 2\mathcal{D}_{\boldsymbol{h}}(\boldsymbol{P})_{ijk} - \Omega X_{ijk} + \sigma Y_{ijk} - 2h_{k[i}W_{j]} - 2K_{k[i}Z_{j]} \\ 2D_{[i}Z_{j]} - 2K_{k[i}P_{j]}{}^k + (D^k\Omega)Y_{ijk} \\ 2D_{[i}W_{j]} - 2L_{[i}Z_{j]} + 2L_{k[i}P_{j]}{}^k - (D^k\Omega)X_{ijk} \\ \epsilon_{ijk}D^kX^{ij}{}_l + 2\epsilon_{ljk}d_i{}^kP^{ij} + 2d^*_{li}Z^i + \epsilon_{ijk}L^iY^{jk}{}_l \\ \epsilon_{ijk}D^kY^{ij}{}_l \end{pmatrix}, \tag{7.2.12}$$

where all index lowering and raising of indices is performed with respect to the metric $\boldsymbol{h}$ and its inverse.

Remark 53. Note that we are now using the fact that, by the auxiliary equation $\mathrm{tr}_{\mathring{\boldsymbol{h}}}\boldsymbol{P} = 0$, $P_{ij} \in \mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})$ and hence the first component of $\mathcal{Q}_w$ can indeed be considered to act on $\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}})$.

The auxiliary equations for $\sigma, s, L_{ij}, K_{ij}$, along with equations (7.2.10a)–(7.2.10e) are then equivalent to the following inhomogeneous equation

$$\mathcal{Q}_w \begin{pmatrix} P_{ij} \\ Z_i \\ W_i \\ X_{ijk} \\ Y_{ijk} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} \mathscr{H}^{(2)}_w(\boldsymbol{Q}) + \mathscr{H}^{(3)}_w(X) \end{pmatrix}, \tag{7.2.13}$$

where $\mathscr{H}^{(2)}_w(\cdot)$, $\mathscr{H}^{(3)}_w(\cdot)$ are the following first-order linear differential operators

$$\mathscr{H}^{(2)}_w(\boldsymbol{Q}) = \begin{pmatrix} -2D_{[i}\Omega U_{j]k} + 2(D_l\Omega)h_{k[i}U_{j]}{}^l \\ 0 \\ 0 \\ 2\epsilon_{ljk}L_i{}^kU^{ij} \\ 2\epsilon_{ljk}K_i{}^kU_{ij} \end{pmatrix}, \qquad \mathscr{H}^{(3)}_w(X) = \begin{pmatrix} 0 \\ \epsilon_{ijk}\Omega D^k X \\ -\epsilon_{ijk}\sigma D^k X \\ 2K_{li}D^iX \\ 2D_lX \end{pmatrix}.$$

As in previous chapters, the sufficiency argument will rely on elliptic properties of the integrability conditions. In the present case we have the following:

Lemma 14. The operator $\mathcal{Q}_w$ is a linear overdetermined elliptic operator in the zero quantities $P_{ij}, Z_i, W_i, X_{ijk}, Y_{ijk}$. Hence, $\mathcal{Q}^*_w \circ \mathcal{Q}_w$ is second order elliptic. [6]

Proof. The principal part of $\mathcal{Q}_w$ is equivalent to

$$\begin{pmatrix} \mathring{\mathcal{D}} & 0 & 0 & 0 & 0 \\ 0 & \mathring{\delta}\oplus d & 0 & 0 & 0 \\ 0 & 0 & \mathring{\delta}\oplus d & 0 & 0 \\ 0 & 0 & 0 & \mathcal{K}_{\boldsymbol{h}} & 0 \\ 0 & 0 & 0 & 0 & \mathcal{K}_{\boldsymbol{h}} \end{pmatrix} \begin{pmatrix} P_{ij} \\ Z_i \\ W_i \\ X_{ijk} \\ Y_{ijk} \end{pmatrix}$$

where here $d : \Lambda^1(\mathcal{S}) \to \Lambda^2(\mathcal{S})$ is the exterior derivative acting on $1-$forms, and $\mathcal{K}_{\boldsymbol{h}} : \mathcal{J}(\mathcal{S}) \to \mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}}) \oplus \Lambda^1(\mathcal{S})$, acts as

$$\mathcal{K}_{\boldsymbol{h}}(\boldsymbol{J}) = \begin{pmatrix} \mathring{\mathcal{D}}^*(\boldsymbol{J})_{ij} \\ \epsilon_{ljk}D^kJ^{lj}{}_i \end{pmatrix}$$

—see Chapter 4. Note that we have placed $\mathring{\mathcal{D}}$ rather than $\mathcal{D}_{\boldsymbol{h}}$ in the symbol map (the component acting on P_{ij}) —we can do so since $\mathring{\mathcal{D}}(\boldsymbol{P})_{ijk}$ and $\mathcal{D}_{\boldsymbol{h}}(\boldsymbol{P})_{ijk}$ differ only by terms which are algebraic in P_{ij} and which therefore do not affect the principal part.

Each of the operators in the diagonal entries is overdetermined elliptic on the relevant spaces, as previously discussed —in particular, for $\mathcal{K}_{\boldsymbol{h}}$, see Lemma 6. Hence, $\mathcal{Q}_u$ is overdetermined elliptic. By standard theory (see Section 2.3.1), the operator $\mathcal{Q}^*_w \circ \mathcal{Q}_w$ from

$$\mathscr{S}^2_0(\mathcal{S};\mathring{\boldsymbol{h}}) \times \Lambda^1(\mathcal{S}) \times \Lambda^1(\mathcal{S}) \times \mathcal{J}(\mathcal{S}) \times \mathcal{J}(\mathcal{S})$$

to itself is then determined elliptic —indeed, its principal part is given by

$$\begin{pmatrix} \mathring{\mathcal{D}}^*\circ\mathring{\mathcal{D}} & 0 & 0 & 0 & 0 \\ 0 & \mathring{\Delta}_H & 0 & 0 & 0 \\ 0 & 0 & \mathring{\Delta}_H & 0 & 0 \\ 0 & 0 & 0 & \mathcal{K}^*_{\boldsymbol{h}}\circ\mathcal{K}_{\boldsymbol{h}} & 0 \\ 0 & 0 & 0 & 0 & \mathcal{K}^*_{\boldsymbol{h}}\circ\mathcal{K}_{\boldsymbol{h}} \end{pmatrix} \begin{pmatrix} P_{ij} \\ Z_i \\ W_i \\ X_{ijk} \\ Y_{ijk} \end{pmatrix}$$

which is manifestly elliptic. □

In addition to the integrability conditions expressed in equation (7.2.13), we have from (7.1.2g)

[6] Here, we choose to compute the L^2-adjoint with respect to the background metric, $\mathring{h}_{ij}$.

and $\Lambda_i = 0$ that

$$\Delta_Y(\boldsymbol{Q})_j = \mathscr{H}_w^{(4)}(\boldsymbol{X}, \boldsymbol{Y})_j \equiv X_j{}^i{}_i + K_{ik}Y_j{}^{ik} - K_{jk}Y^{ik}{}_i - KY_j{}^i{}_i, \tag{7.2.14}$$

while the integrability condition (7.1.2d), along with the auxiliary equation $\bar{X}_i = 0$, imply

$$\Delta X = \mathscr{H}_w^{(1)}(\boldsymbol{P}, \boldsymbol{X}, \boldsymbol{Y}) \equiv \tfrac{1}{2}\epsilon_{ijl}K_k{}^l X^{ijk} - d^*_{ij}P^{ij} - \tfrac{1}{2}\epsilon_{ijl}L_k{}^l Y^{ijk}. \tag{7.2.15}$$

For what follows, it will prove convenient to also introduce the operator Υ_w mapping

$$\mathscr{C}(\mathcal{S}) \times \mathscr{S}_0^2(\mathcal{S}; \mathring{\boldsymbol{h}}) \times \Lambda^1(\mathcal{S}) \times \Lambda^1(\mathcal{S}) \times \mathcal{J}(\mathcal{S}) \times \mathcal{J}(\mathcal{S}) \times \Lambda^1(\mathcal{S})$$

to itself, which we express matricially as follows

$$\Upsilon_w \begin{pmatrix} X \\ P_{ij} \\ Z_i \\ W_i \\ X_{ijk} \\ Y_{ijk} \\ Q_i \end{pmatrix} \equiv \left(\begin{array}{c:c:c} \Delta & -\mathscr{H}_w^{(1)}(\cdot) & 0 \\ \hdashline & & \\ \tilde{\mathscr{H}}_w^{(3)}(\cdot) & \mathcal{Q}_w^* \circ \mathcal{Q}_w(\cdot) & \tilde{\mathscr{H}}_w^{(2)}(\cdot) \\ & & \\ \hdashline 0 & -\mathscr{H}_w^{(4)}(\cdot) & \Delta_Y \end{array}\right) \begin{pmatrix} X \\ P_{ij} \\ Z_i \\ W_i \\ X_{ijk} \\ Y_{ijk} \\ Q_i \end{pmatrix}.$$

with $\tilde{\mathscr{H}}_w^{(2)} \equiv -\mathcal{Q}_w^* \circ (\boldsymbol{0} \oplus \mathscr{H}_w^{(2)})$ and $\tilde{\mathscr{H}}_w^{(3)} \equiv -\mathcal{Q}_w^* \circ (\boldsymbol{0} \oplus \mathscr{H}_w^{(3)})$. Denoting

$$z \equiv (X,\ P_{ij},\ Z_i,\ W_i,\ X_{ijk},\ Y_{ijk},\ Q_i),$$

equations (7.2.13), (7.2.14) and (7.2.15) are expressed collectively simply by the equation

$$\Upsilon_w(z) = 0. \tag{7.2.16}$$

The result of the above discussion is as follows:

Lemma 15. Let

$$w \equiv (\Omega,\ \sigma,\ s,\ L_i,\ L_{ij},\ K_{ij},\ d^*_{ij},\ d_{ij},\ h_{ij})$$

be a candidate solution, then the resulting zero quantities

$$z = z(w) \equiv (X,\ P_{ij},\ Z_i,\ W_i,\ X,\ X_{ijk},\ Y_{ijk},\ Q_i)$$

solve the *linear second-order elliptic* equation $\Upsilon_w(z) = 0$.

Proof. That z solves $\Upsilon_w(z) = 0$ is clear from the above discussion. To see that Υ_w is elliptic, note first that $\mathscr{H}_w^{(1)}, \mathscr{H}_w^{(4)}$ are first-order operators of $X,\ Q_i$, respectively, and therefore do not enter the principal part. On the other hand, $\mathscr{H}_w^{(2)}, \mathscr{H}_w^{(3)}$ are second-order operators. However, given z in the kernel of $\sigma_{\boldsymbol{\xi}}[\Upsilon_w]$, we see that ellipticity of the first and last components —i.e. of $\Delta,\ \Delta_Y$— imply that $X = Q_i = 0$. Substituting back into $\sigma_{\boldsymbol{\xi}}[\Upsilon_w](z) = 0$ we obtain

$$\sigma_{\boldsymbol{\xi}}[\mathcal{Q}_w^* \circ \mathcal{Q}_w](\boldsymbol{P}, \boldsymbol{Z}, \boldsymbol{W}, \boldsymbol{X}, \boldsymbol{Y}) = 0.$$

Since $\mathcal{Q}_w^* \circ \mathcal{Q}_w$ is elliptic, we find $P_{ij} = Z_i = W_i = X_{ijk} = Y_{ijk} = 0$, and hence Υ_w is second-order elliptic, as claimed. □

Remark 54. Notice that for our choice of background solution, $\mathring{w}$, the operators $\mathscr{H}^{(1)}_{\mathring{w}}$, $\mathscr{H}^{(2)}_{\mathring{w}}$ trivialise. Indeed,

$$\begin{aligned}\mathscr{H}^{(1)}_{w}(\boldsymbol{P},\boldsymbol{X},\boldsymbol{Y}) &= \tfrac{1}{2}\mathring{\epsilon}_{ijl}\mathring{K}_k{}^l X^{ijk} - \mathring{d}^*_{ij}P^{ij} - \tfrac{1}{2}\mathring{\epsilon}_{ijl}\mathring{L}_k{}^l Y^{ijk} \\ &= \tfrac{1}{2}\mathring{K}\mathring{\epsilon}_{ijk}X^{ijk} - \tfrac{\lambda}{12}\mathring{\epsilon}_{ijk}Y^{ijk} \\ &= 0\end{aligned}$$

using the fact that $\mathring{d}^*_{ij} = 0$ and that $X_{[ijk]} = Y_{[ijk]} = 0$ by Jacobi symmetry —i.e. since $X_{ijk},\ Y_{ijk} \in \mathcal{J}(\mathcal{S})$. Similarly, $\mathscr{H}^{(2)}_{\mathring{w}}$ trivialises simply by noting that $d\mathring{\Omega} = 0$ and using the fact that $\mathring{L}_{ij}$, $\mathring{K}_{ij}$ are pure-trace. It follows by linearity of $\mathcal{Q}^*_{\mathring{w}}$ that $\tilde{\mathscr{H}}^{(2)}_{\mathring{w}}$ also trivialises. Hence, $\Upsilon_{\mathring{w}}$ (thought of a block matrix with blocks demarcated by the dotted lines) in lower-triangular. The semi-decoupling of the equations reflected in this allows for a particularly straightforward analysis of its kernel.

We now aim to show that the only solution is to $\Upsilon_w(z)$ for w sufficiently close to $\mathring{w}$, is the trivial solution $z = 0$. Having done so, we will have shown that our candidate solution w is indeed a solution to the CCEs. By analogy with previous chapters, we proceed as follows: first we show that $\Upsilon_{\mathring{w}}$ is injective and then we appeal to the stability property of kernels of elliptic operators to ensure that Υ_w is also injective for w close to $\mathring{w}$. To prove the injectivity of $\Upsilon_{\mathring{w}}$ we first show injectivity of $\mathcal{Q}^*_{\mathring{w}} \circ \mathcal{Q}_{\mathring{w}}$:

Lemma 16. Given a conformally rigid hyperbolic solution, $\mathring{w}$, to the CCEs on a closed $\mathcal{S}$, the operator $\mathcal{Q}_{\mathring{w}}$ is injective on the space of smooth sections, and hence so is $\mathcal{Q}^*_{\mathring{w}} \circ \mathcal{Q}_{\mathring{w}}$.

Proof. Evaluated at $w = \mathring{w}$, we have

$$\mathcal{Q}_{\mathring{u}}\begin{pmatrix} P_{ij} \\ Z_i \\ W_i \\ X_{ijk} \\ Y_{ijk}\end{pmatrix} = \begin{pmatrix} \mathring{\delta}(\boldsymbol{Z}) \\ \mathring{\delta}(\boldsymbol{W}) \\ \mathring{\mathcal{D}}^*(\boldsymbol{X})_{ij} \\ \mathring{\mathcal{D}}^*(\boldsymbol{Y})_{ij}\end{pmatrix} \oplus \begin{pmatrix} \mathring{\epsilon}_{lik}\mathring{D}^k P_j{}^i - W^i\mathring{\epsilon}_{jli} + \tfrac{1}{2}X^{ik}{}_j\mathring{\epsilon}_{lik} - \tfrac{1}{3}\mathring{K}Z^i\mathring{\epsilon}_{lij} \\ 2\mathring{D}_{[i}Z_{j]} \\ 2\mathring{D}_{[i}W_{j]} \\ \mathring{\epsilon}_{ijk}\mathring{D}^k X^{ij}{}_l \\ \mathring{\epsilon}_{ijk}\mathring{D}^k Y^{ij}{}_l \end{pmatrix}.$$

From the vanishing of the first Betti number of $\mathcal{S}$ there are no harmonic 1-forms (c.f. Hodge's Theorem) so it follows immediately that $W_i = 0$ and $Z_i = 0$. Moreover, the last two components are precisely the statement that

$$\mathring{\mathcal{K}}(\boldsymbol{X}) = \mathring{\mathcal{K}}(\boldsymbol{Y}) = 0.$$

Recall from Chapter 4 that $\mathring{\mathcal{K}}$ has trivial kernel —see Proposition 7. Hence, $X_{ijk} = Y_{ijk} = 0$. Finally, substituting into the first component, we see that $\epsilon_{lik}\mathring{D}^k P_j{}^i = 0$, or equivalently,

$$\mathring{\mathcal{D}}(\boldsymbol{P})_{ijk} = 0.$$

That is to say, P_{ij} is a Codazzi tensor. Recall however that by virtue of the auxiliary equation, $P_{ij} \in \mathscr{S}^2_0(\mathcal{S},\mathring{\boldsymbol{h}})$ and so $P_{ij} = 0$ since $(\mathcal{S},\mathring{\boldsymbol{h}})$ admits no tracefree Codazzi tensors (i.e. conformal rigidity). Hence we see that $\mathcal{Q}_{\mathring{w}}$ is injective. Injectivity of $\mathcal{Q}^*_{\mathring{w}} \circ \mathcal{Q}_{\mathring{w}}$ follows by a simple integration by parts argument. □

Proposition 22. Given a conformally rigid hyperbolic solution, $\mathring{w}$, to the CCEs on a closed $\mathcal{S}$, the operator $\Upsilon_{\mathring{w}}$ is injective.

Proof. Recall from Remark 54 that $\mathscr{H}^{(1)}_{\mathring{w}}$ trivialises, and so the first component of $\Upsilon_{\mathring{w}}(z) = 0$ implies that X is harmonic with respect to $\mathring{h}_{ij}$ —i.e. that

$$\mathring{\Delta} X = 0$$

—and therefore X is constant over $\mathcal{S}$. However, recall that $X \equiv \mathring{\delta}(\boldsymbol{\varphi})$ and therefore integrates to zero by the divergence theorem, so we see that $X = 0$. Substituting back into $\Upsilon_{\mathring{w}}(z) = 0$ we have

$$\left(\begin{array}{c:c} & 0 \\ & 0 \\ \mathcal{Q}^*_{\mathring{w}} \circ \mathcal{Q}_{\mathring{w}}(\cdot) & 0 \\ & 0 \\ & 0 \\ \hdashline -\mathscr{H}^{(4)}_{\mathring{w}}(\cdot) & \mathring{\Delta}_Y \end{array}\right) \begin{pmatrix} P_{ij} \\ Z_i \\ W_i \\ X_{ijk} \\ Y_{ijk} \\ Q_i \end{pmatrix} = 0,$$

where we are using the fact $\tilde{\mathscr{H}}^{(2)}_{\mathring{w}}$ trivialises —see Remark 54, once more. Hence, the fields P_{ij}, Z_i, W_i, X_{ijk}, Y_{ijk} lie in the kernel of $\mathcal{Q}^*_{\mathring{w}} \circ \mathcal{Q}_{\mathring{w}}$ and therefore vanish identically by the previous proposition. Substituting

$$P_{ij} = Z_i = W_i = X_{ijk} = Y_{ijk} = 0$$

into the above equation we find

$$\mathring{\Delta}_Y Q_i = 0.$$

Hence, $Q_i = 0$ —recall from Chapter 4 that $\mathring{\Delta}_Y$ is positive-definite for a hyperbolic metric, $\mathring{h}_{ij}$. The operator $\Upsilon_{\mathring{w}}$ is therefore injective, as claimed. □

Now that we have shown triviality of the kernel of $\Upsilon_{\mathring{w}}$ at the background solution $w = \mathring{w}$, we now aim to extend the result to operators Υ_w where w is a candidate solutions close to $\mathring{w}$.

Proposition 23. There exists $\delta > 0$ such that, given free data $u \equiv (\varphi,\ \theta,\ \phi,\ \psi_{ij},\ \psi^*_{ij})$ satisfying

$$\|u - \mathring{u}\|_{\mathcal{X}^k} < \delta$$

then the resulting candidate solution

$$w(u) = w(\nu(u)) \equiv (\Omega,\ \sigma,\ s,\ L_i,\ L_{ij},\ K_{ij},\ d^*_{ij},\ d_{ij},\ h_{ij})$$

solves the CCEs with cosmological constant given by

$$\lambda \equiv 6\Omega s + 3\sigma^2 - 3\|d\Omega\|^2_{\boldsymbol{h}}. \tag{7.2.17}$$

Proof. First note that the fields comprising w are, by construction, at least H^{k-2} ($k \geq 4$) and that the coefficients of Υ_w are at most second derivatives of w. The map

$$w \mapsto \Upsilon_w$$

is therefore Lipschitz continuous as a map from H^2 to $B(H^2, L^2)$ and so (see Remark 23) a modification of the proof of Proposition 8 shows that there exists $\epsilon > 0$ such that, if

$$\|w - \mathring{w}\|_{H^2} < \epsilon$$

then Υ_w is injective. Recall that the map $\nu : H^{k-2} \to H^k$ is continuous as is the map $(u,v) \mapsto w(u,v)$; note that the image consists of sections which are at least $H^{k-2} \subset H^2$. Hence there exists some δ for which $\Upsilon_{\mathring{w}(u)}$ is injective whenever

$$\|u - \mathring{u}\|_{\mathcal{X}^k} < \delta.$$

Now, given such a candidate solution, $w(u)$, the resulting zero quantities

$$z(w) = (X,\ P_{ij},\ Z_i,\ W_i,\ X_{ijk},\ Y_{ijk},\ Q_i)$$

satisfy $\Upsilon_{w(u)}(z(w)) = 0$ and hence vanish by injectivity of $\Upsilon_{w(u)}$. Then, it follows from (7.1.2h) that

$$D_i(6\Omega s + 3\sigma^2 - 3\|d\Omega\|^2_{\boldsymbol{h}}) = 6\Omega W_i + 6\sigma Z_i - 6D^j\Omega P_{ij} = 0,$$

and so the expression

$$6\Omega s + 3\sigma^2 - 3\|d\Omega\|^2_{\boldsymbol{h}}$$

is constant on $\mathcal{S}$. Defining this to be the cosmological constant, then the algebraic constraint $A = 0$ is trivially satisfied. By choosing δ sufficiently small, it is clear that λ has the same sign as $\mathring{\lambda}$. $\square$

Combining Propositions 20 and 23 establishes parts (i) and (ii) of Theorem 7.

7.2.3 Relating the solutions of the ECEs and CCEs

Here we aim to relate the above-described solutions to those constructed in Chapter 4. Since it is not much more involved, we consider a slightly more general picture in which we allow for the background solution to correspond to a physical background initial data set by a non-trivial conformal rescaling:

$$\mathring{h}_{ij} = \mathring{\Omega}^2 \mathring{\tilde{h}}_{ij}.$$

Recall that in the application considered above, we took $\mathring{\Omega} = 1$ so that the physical and unphysical background metrics are the same.

Given a solution of the CCEs

$$(\Omega,\ s,\ \sigma,\ L_i,\ L_{ij},\ K_{ij}, d^*_{ij},\ d_{ij},\ h_{ij}),$$

recall from Section 3.2.2 that the corresponding physical solution to the ECEs is given by

$$\begin{aligned}
\tilde{K}_{ij} &= \Omega^{-1}K_{ij} - \sigma\Omega^{-2}h_{ij} \\
\bar{S}_{ij} &= \Omega d^*_{ij}, \\
S_{ij} &= \Omega d_{ij}, \\
\tilde{h}_{ij} &= \Omega^{-2}h_{ij}.
\end{aligned}$$

We would like to show that such a solution can be realised in the form considered in Chapter 4, for a particular choice of physical free data $(\phi, \bar{T}_{ij}, T_{ij})$. In other words we would like to write

$$\begin{aligned}
\tilde{K}_{ij} &= \tilde{\chi}_{ij} + \tfrac{1}{3}\tilde{\phi}\mathring{\tilde{h}}_{ij}, \\
\bar{S}_{ij} &= \Pi_{\tilde{\boldsymbol{h}}}(\mathring{\tilde{L}}(\bar{\boldsymbol{X}}) + \bar{\boldsymbol{T}})_{ij}, \\
S_{ij} &= \Pi_{\tilde{\boldsymbol{h}}}(\mathring{\tilde{L}}(\boldsymbol{X}) + \boldsymbol{T})_{ij}
\end{aligned}$$

for some $\tilde{\chi}_{ij}$ and $\bar{T}_{ij}, T_{ij} \in \mathscr{S}_{TT}(\mathcal{S}; \mathring{\tilde{\boldsymbol{h}}})$. For the first we simply take

$$\tilde{\phi} = \Omega\phi - \sigma(\mathrm{tr}_{\mathring{\boldsymbol{h}}}\boldsymbol{h})$$

and rearrange to determine $\tilde{\chi}_{ij}$. Now consider the third (the second can of course be treated in the same way); we want to satisfy

$$\Pi_{\tilde{\boldsymbol{h}}}(\mathring{\tilde{L}}(\boldsymbol{X}) + \boldsymbol{T}) = \Omega\Pi_{\boldsymbol{h}}(\mathring{L}(\boldsymbol{u}) + \boldsymbol{\psi}).$$

Noting that the projection operator, $\Pi_{\boldsymbol{h}}$, is invariant under conformal changes of the metric, this is equivalent to solving

$$\mathring{\tilde{L}}(\boldsymbol{X})_{ij} + T_{ij} = \Omega\mathring{L}(\boldsymbol{u}) + \Omega\psi_{ij}. \tag{7.2.18}$$

Taking the $\mathring{\tilde{\boldsymbol{h}}}-$divergence of the right-hand-side,

$$\begin{aligned}
\mathring{\tilde{\delta}}(\Omega\mathring{L}(\boldsymbol{u}))_i + \mathring{\tilde{\delta}}(\Omega\psi)_i &= \mathring{\tilde{\delta}}(\Omega\mathring{L}(\boldsymbol{u}))_i + \mathring{\tilde{\delta}}(\mathring{\Omega}^{-1}\Omega(\mathring{\Omega}\psi))_i \\
&= \mathring{\tilde{\delta}}(\Omega\mathring{L}(\boldsymbol{u}))_i + \mathring{\Omega}d(\mathring{\Omega}^{-1}\Omega)^j\psi_{ij} + \mathring{\Omega}^{-1}\Omega\mathring{\tilde{\delta}}(\mathring{\Omega}\psi)_i \\
&= \mathring{\tilde{\delta}}(\Omega\mathring{L}(\boldsymbol{u}))_i + \mathring{\Omega}d(\mathring{\Omega}^{-1}\Omega)^j\psi_{ij} + \mathring{\Omega}^{-1}\Omega\mathring{\delta}(\psi)_i \\
&= \mathring{\tilde{\delta}}(\Omega\mathring{L}(\boldsymbol{u}))_i + \mathring{\Omega}d(\mathring{\Omega}^{-1}\Omega)^j\psi_{ij},
\end{aligned}$$

where contractions are performed using $\mathring{\tilde{h}}^{ij}$. The penultimate line follows from the transformation law for the divergence operator on $2-$tensors and the final line follows from the fact that $\psi_{ij} \in \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})$. Hence, if we solve

$$\mathring{\tilde{\delta}} \circ \mathring{\tilde{L}}(\boldsymbol{X})_i = \mathring{\tilde{\delta}}(\Omega\mathring{L}(\boldsymbol{u}))_i + \mathring{\Omega}d(\mathring{\Omega}^{-1}\Omega)^j\psi_{ij}$$

for X_i, and then define

$$T_{ij} \equiv -\mathring{\tilde{L}}(\boldsymbol{X})_{ij} + \Omega\mathring{L}(\boldsymbol{u}) + \Omega\psi_{ij},$$

then, by construction, $T_{ij} \in \mathscr{S}_{TT}(\mathcal{S}; \mathring{\tilde{\boldsymbol{h}}})$ and (7.2.18) will be satisfied. In the case considered here we have $\mathring{h}_{ij} = \mathring{\tilde{h}}_{ij}$ (since $\mathring{\Omega} \equiv 1$) and hence the above reduces to solving

$$\mathring{\delta} \circ \mathring{L}(\boldsymbol{X})_i = \mathring{\delta}(\Omega\mathring{L}(\boldsymbol{u}))_i + (d\Omega)^j\psi_{ij}$$

whereupon T_{ij} is given by

$$T_{ij} \equiv -\mathring{L}(\boldsymbol{X})_{ij} + \Omega\mathring{L}(\boldsymbol{u}) + \Omega\psi_{ij}.$$

Conclusion (iii) of Theorem 7 then follows. Note that in order to determine the physical free data, according to the above formulae, we need to have first solved the CCEs to determine $\Omega, \sigma, u_i^*, u_i, h_{ij}$. This is analogous to the "Conformal Method" —see Section 1.3. Moreover, in relating the electric (resp. magnetic) and *rescaled* electric (resp. magnetic) parts, the free and determined fields become mixed —one needs both ψ_{ij} and u_i to determine T_{ij}, for instance.

7.3 Towards an elliptic BVP for the CCEs

The purpose of this section is to discuss some aspects of the boundary-value-problem (BVP) for the CCEs as an elliptic system, as described above. The discussion is heuristic.

In order to extend the method to the case of compact $\mathcal{S}$ with boundary, the elliptic auxiliary equations presented in this chapter must by complemented by the prescription on $\partial\mathcal{S}$ of suitable *elliptic boundary conditions* for the fields

$$v = (\Omega, \sigma_i, \sigma, s, L_i, L_{ij}, K_{ij}, d^*_{ij}, d_{ij}, h_{ij})$$

—i.e. boundary conditions for which the corresponding BVP is elliptic in the sense of Lopatinski–Shapiro (see e.g. Chapter II, section 11 of [80]). Of course, we must prescribe $\Omega = 0$ on $\partial\mathcal{S}$ in order that the boundary may be identified with conformal infinity of the physical solution. The boundary values of the remaining fields are constrained, however, to satisfy the projections (onto $\partial\mathcal{S}$) of the CCE equations, $\Xi = 0$ —i.e. the $2+1$ constraint equations implied by the CCEs on $\partial\mathcal{S}$. It is reasonable to expect that $v|_{\partial\mathcal{S}}$ should be constructible algebraically from a smaller "generating set" of fields, analogous to Friedrich's construction of asymptotic initial data —see [36].

The presence of a boundary not only complicates the analysis of the linearised (auxiliary) equations, but also the "sufficiency" argument —if one tries to repeat the analysis of Section 4 then there are boundary terms which need to taken into account. Note however that, in prescribing $v|_{\partial\mathcal{S}}$ subject to the $2+1$ constraint equations, certain components of the zero quantities will vanish by construction, leading to simplifications of the boundary terms. It is possible, however, that tractability of the analysis will require further restrictions on the choice of boundary values.

It is also reasonable to expect that, given appropriate boundary conditions, conformal Killing fields and Codazzi tensors (and other obstructions which may arise) may be eliminated from the kernel and cokernel of $D_u\widetilde{\Xi}$. For instance, it is shown in [81] that if condition (a) (resp. (b)), below, is satisfied then there are no conformal Killing fields on $\mathcal{S}$ that are tangential (resp. orthogonal) to the boundary $\partial\mathcal{S}$:

(a) The Ricci tensor is negative-definite on $\mathcal{S}$ and the extrinsic curvature (with outward-pointing normal) of $\partial\mathcal{S} \subset \mathcal{S}$ is positive semi-definite;

(b) The Ricci tensor is negative-definite on $\mathcal{S}$ and the mean extrinsic curvature (with outward-pointing normal) is positive.

As a first approach one could consider as a background geometry the unit hyperboloid in Minkowski space, which may be conformally compactified to give the flat unit disk. Since this admits conformal Killing fields that are tangential to $\partial\mathcal{S}$ (the rotational Killing fields) as well as one that is orthogonal to $\partial\mathcal{S}$ (the dilation $r\partial_r$) —in addition to Killing fields (e.g. the translations) which are neither tangential nor orthogonal to $\partial\mathcal{S}$— we cannot hope to eliminate conformal Killing fields from the kernel/cokernel simply by imposing that either the tangential or normal components vanish on $\partial\mathcal{S}$.

Another possibility is to exclude certain obstructions by considering their decay at infinity on the physical manifold. This could possibly be effected on the unphysical manifold $\mathcal{S}$ through the use of weighted Sobolev spaces.

A third possibility is to impose inadmissible "Killing data" on $\partial\mathcal{S}$: it is shown in [56] that to uniquely determine a conformal Killing vector on $\mathcal{S}$, it is sufficient to prescribe the fields

$$V_i,\ D_{[i}V_{j]},\ \delta(\boldsymbol{V}),\ D_i\delta(\boldsymbol{V})$$

(the "Killing data") at any $p \in \mathcal{S}$. Moreover, those fields are required to satisfy an algebraic condition at $p \in \mathcal{S}$ in order for the conformal Killing vector equation to be integrable in a neighbourhood of the point p. If we were to somehow prescribe (via boundary conditions) the fields $V_i,\ D_{[i}V_{j]},\ \delta(\boldsymbol{V}),\ D_i\delta(\boldsymbol{V})$ such that the integrability condition fails to be satisfied at some $p \in \mathcal{S}$, then there can be no conformal Killing vector field in any neighbourhood of p with the given Killing data, and hence no global conformal Killing vector field. Whether or not there is enough freedom to prescribe all components of the Killing data requires a closer inspection of the "2+1" constraint equations.

7.3.1 Fixing the cosmological constant

One of the deficits of the method described in the previous sections is that the cosmological constant is not fixed at the outset, but rather it must be fixed (algebraically) for each solution of the CCEs so constructed. In the BVP, one approach around this problem is to attempt to fix the cosmological constant by prescribing Ω, σ, s on $\partial\mathcal{S}$ such that

$$(6\Omega s + 3\sigma^2 - 3\|d\Omega\|^2)|_{\partial\mathcal{S}} = \lambda.$$

for a given λ, fixed at the outset. Note that the term $\|d\Omega\|^2$ contains normal derivatives of Ω (that is to say, normal to $\partial\mathcal{S} \subset \mathcal{S}$) which cannot be prescribed directly —recall that we need to prescribe $\Omega|_{\partial\mathcal{S}} = 0$ so that $\partial\mathcal{S}$ can be identified with conformal infinity of the physical initial data set. One method around this problem is to first perform a first-order reduction of the conformal factor equation $P_{ij} = 0$; one makes a replacement $(d\Omega)_i \mapsto \sigma_i$, and considers instead following first order equation in σ_i

$$\bar{P}_{ij} \equiv D_i\sigma_j + \Omega L_{ij} - \sigma K_{ij} - sh_{ij} = 0.$$

Note now that the zero quantity $\bar{P}_{ij}$ is not a priori symmetric. To complete the system, one then imposes the vanishing of the additional zero quantity

$$P_i \equiv D_i\Omega - \sigma_i,$$

which encodes the definition of σ_i as the gradient of Ω. The zero quantity Q_i is subject to the following integrability condition

$$D_{[i}P_{j]} = -\bar{P}_{[ij]}, \tag{7.3.1}$$

or equivalently,

$$\mathrm{curl}_{\boldsymbol{h}}(\boldsymbol{P})_i = -\epsilon_i{}^{jk}\bar{P}_{jk}.$$

For $\bar{P}_{ij}$, we have the following integrability condition

$$\epsilon_{lik}D^k\bar{P}^i{}_j = \epsilon_{jli}W^i - \tfrac{1}{2}\Omega\epsilon_{lik}X^{ik}{}_j + \epsilon_{lik}K_j{}^kZ^i + \tfrac{1}{2}\sigma\epsilon_{lik}Y^{ik}{}_j + \epsilon_{jlk}\sigma^iU_i{}^k + \epsilon_{lik}\sigma^iU_j{}^k. \tag{7.3.2}$$

On the other hand, the following auxiliary equations for Ω and σ_i are clearly elliptic

$$0 = \delta(\boldsymbol{P}) \equiv \Delta\Omega - D^i\sigma_i, \tag{7.3.3a}$$

$$0 = D^i\bar{P}_{ij} \sim \Delta\sigma_i. \tag{7.3.3b}$$

Performing the same replacement replacement $(d\Omega)_i \mapsto \sigma_i$ in the remaining CCEs, it is clear that the remaining zero quantities will also be subject to a family of integrability conditions, and that the principal parts of these integrability conditions will be identical to before —the two systems differ only by the addition of linear algebraic expressions in P_i. It is clear moreover that these slightly modified CCEs enjoy the same elliptic reduction procedure as before. In the previous analysis, recall that the presence of elliptic structures in the joint auxiliary-integrability system, Υ, was crucial for the sufficiency argument. Here, we see again that the auxiliary-integrability system admits an elliptic reduction for the relevant zero quantities. Since the auxiliary equations for the fields $\sigma, s, L_i, L_{ij}, K_{ij}, d_{ij}, d^*_{ij}, h_{ij}$ (in terms of the zero quantities) are the same as before, as are the principal parts of the corresponding integrability relations, we need only consider the equations for $P_i,\ \bar{P}_{ij}$. In the case of P_i, we have $\mathrm{curl}(\boldsymbol{P})_i = \delta(\boldsymbol{P}) = 0$, and hence the elliptic equation

$$\Delta_H P_i = 0$$

holds —this mirrors the treatment of the zero quantities $W_i,\ Z_i$, earlier. On the other hand, for $\bar{P}_{ij}$ we find that taking the divergence of equation (7.3.2) and using the auxiliary equation (7.3.3b), the resulting equation has principal part

$$\begin{aligned} D^i(D_i\bar{P}_{jk} - D_j\bar{P}_{ik}) &= \Delta\bar{P}_{jk} - D^iD_j\bar{P}_{ij} \\ &= \Delta\bar{P}_{jk} - D_jD^i\bar{P}_{ij} + r_k{}^l{}_j{}^i\bar{P}_{il} - r_j{}^l\bar{P}_{lk} \\ &= \Delta\bar{P}_{jk} + r_k{}^l{}_j{}^i\bar{P}_{il} - r_j{}^l\bar{P}_{lk}, \end{aligned}$$

which is manifestly elliptic. Hence, we see that the joint auxiliary-integrability system again admits an elliptic reduction, now as a system for the zero quantities

$$(P_i,\ \bar{P}_{ij},\ Z_i,\ W_i,\ X_{ij},\ X_{ijk},\ Y_{ijk},\ \Lambda^*_i,\ \Lambda_i,\ U_{ij}),$$

analogous to that described in Section 7.2.2. A similar sufficiency argument could, in principle, be carried out.

7.4 Concluding remarks

In this chapter, the Friedrich–Butscher method was extended to the full CCE system. To do so, the intrinsic and extrinsic conformal freedom were fixed by prescription of certain components of the tangential-tangential and tangential-normal components of the 4-dimensional Schouten tensor. These components were chosen in order to allow for an elliptic reduction of the CCEs. The framework was then applied once again to conformally rigid hyperbolic data (extended trivially to a solution of the full CCEs); the resulting Theorem 7 can be thought of as the conformal analogue of Theorem 2 from Chapter 4. The solutions furnished by Theorem 7 are, in general, expressed in a non-trivial conformal gauge which must be undone by a conformal transformation to recover the corresponding physical solution.

It is also interesting to note that in the analysis we required injectivity of the Hodge Laplacian —i.e. the vanishing of the first Betti number of $\mathcal{S}$. While this condition is satisfied by the conformally rigid hyperbolic initial data sets, a natural question is whether, in general, the requirement places additional restrictions on the background initial data set than already imposed within the context of the ECEs. Given the correspondence of solutions to the ECEs and the CCEs, it is reasonable to assume that there exists an extension of the Friedrich–Butscher method to the full CCEs which introduces no further obstructions. There are two possibilities: either the method described in this chapter requires modification to avoid the introduction of additional requirements on the background solution, or that there is some deeper connection between the obstructions arising in the methods (as presented) —i.e. the existence of harmonic 1-forms, conformal Killing vector fields and Codazzi tensors etc. This will be investigated elsewhere.

Chapter 8

Approximate KID sets

Of significant interest in the Cauchy problem is the question of under which conditions an initial data set gives rise to a spacetime development with Killing symmetries. This question first arose in the context of linearisation stability, see [8]. These conditions are encoded in the so-called Killing Initial Data (KID) equations, which we saw in Chapter 6 —see e.g. [10, 82] for a discussion of the basic properties of these equations; see also [83]. The KID equations constitute a system of overdetermined equations for a scalar and a vector on the initial hypersurface. The existence of a solution to these equations is equivalent to the existence of a Killing vector in the development of the initial data. The KID equations have a deep connection with the *Arnowit-Deser-Misner (ADM) evolution equations*: the evolution equations can be described as a flow generated by the adjoint linearised constraint map, $D\Phi^*$ (see below) —see e.g. [84] for further details.

In many applications of both physical and mathematical interest it is important to have a way of quantifying how much a give initial data set deviates from stationarity. Ideally, one would like to do this in coordinate-independent manner. One approach to this problem was proposed in [11], in which the notion of an *approximate Killing vector*, as a solution to a fourth-order linear elliptic system arising from the KID equations, was introduced. The so-called *approximate Killing vector equation* has the property that its solution set contains that of the KID equations. The analysis in [11] was restricted to the case of time symmetric asymptotically-Euclidean initial data sets. In particular, it was shown that the kernel of the approximate Killing vector equation is non-trivial, and moreover that, given suitable assumptions on the asymptotics of the initial data set, the solution (termed the *approximate Killing vector*) is unique up to constant rescaling.

This general strategy has also been adapted to the study of Killing spinors —see [85]. These ideas have been used, in turn, to obtain an invariant characterising initial data sets for the Kerr spacetime, see [12, 86], and for the Kerr-Newman spacetime, see [13].

The purpose of this chapter is to extend Dain's result in [11] to the non-time symmetric case. Moreover, we analyse in some detail conformally flat initial data sets as way of obtaining some further insight into Dain's construction. Our main result is Theorem 8 which shows that the approximate Killing vector equation can be solved with the required asymptotic conditions for a large class of asymptotically Euclidean initial data sets. The work of this chapter is based on the paper

- Valiente Kroon, J.A. and Williams, J.L., "Dain s invariant on non-time symmetric initial data sets", *Classical and Quantum Gravity*, 34.12 (2017): 125013.

This chapter is structured as follows: Section 8.1 provides a discussion of the basic properties of the approximate Killing vector equation as introduced by Dain. In particular, Subsection 8.1.1 provides a discussion of the relation between the Einstein constraint equations and the so-called Killing Initial Data (KID) equations; Subsection 8.1.2 provides a detailed discussion of the approximate Killing vector equation in the non-time symmetric setting; Subsection 8.1.3 introduces some useful identities which will be used throughout. Section 8.2 analyses the solvability of the approximate Killing equation on asymptotically Euclidean manifolds: in Subsection 8.2.1 some basic background on weighted Sobolev spaces is given; Subsection 8.2.2 provides a discussion of our main asymptotic decay assumptions and of the asymptotic behaviour of solutions to the KID equations; Subsection 8.2.3 briefly reviews the basic methods to analyse the existence of solutions to elliptic equations on asymptotically Euclidean manifolds; Subsection 8.2.4 contains our main existence results. Finally, Section 8.3 contains a further discussion of the geometric invariant obtained from Dain's construction with particular emphasis on the case of conformally flat initial data sets.

8.1 The approximate Killing vector equation

In this section we introduce the basic objects of our analysis: the vacuum Einstein constraint equations, the Killing initial data equations and the approximate Killing initial data equations.

8.1.1 The Einstein constraints and the KID equations

In this section we will study properties of initial data sets for the vacuum Einstein field equations —that is, triples $(\mathcal{S}, h_{ij}, K_{ij})$ where $\mathcal{S}$ is a 3-dimensional manifold, h_{ij} is a Riemannian metric on $\mathcal{S}$ and K_{ij} is a symmetric rank 2 tensor satisfying the vacuum Einstein constraint equations

$$r + K^2 - K_{ij}K^{ij} = 0, \tag{8.1.1a}$$

$$D^j K_{ij} - D_i K = 0. \tag{8.1.1b}$$

In the following we will be particularly interested in initial data sets $(\mathcal{S}, h_{ij}, K_{ij})$ whose development has a Killing vector. The conditions for this to be case are identified in the following:

Proposition 24. Let $(\mathcal{S}, h_{ij}, K_{ij})$ denote an initial data set for the vacuum Einstein field equations. If there exists and scalar field N and a vector field Y^i over $\mathcal{S}$ satisfying the equations

$$L_{ij} \equiv NK_{ij} + D_{(i}Y_{j)} = 0, \tag{8.1.2a}$$

$$\begin{aligned} M_{ij} \equiv\ & Y^k D_k K_{ij} + D_i Y^k K_{kj} + D_j Y^k K_{ik} + D_i D_j N \\ & -N(r_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) = 0, \end{aligned} \tag{8.1.2b}$$

then the development of the initial data is endowed with a Killing vector.

A proof of this result can be found in e.g. [82] —see also [10].

Remark 55. The pair (N, Y^i) is called a *Killing initial data set (KID)* and equations (8.1.2a)-(8.1.2b) are known as the *KID equations*.

It is interesting to note that Killing initial data for conformally rescaled vacuum spacetimes has been analysed in [87, 88], with applications to the characterisation of Kerr-de Sitter-like spacetimes in [89].

8.1.2 Basic properties of the approximate Killing vector equation

In the following, denote by $\mathscr{M}_2$, $\mathscr{S}^2$, $\mathscr{X}$ and $\mathscr{C}$ the spaces of Riemannian metrics, symmetric 2-tensors, vectors and scalar functions on the 3-dimensional manifold $\mathcal{S}$, respectively. It is convenient to write the Einstein constraint equations (8.1.1a) and (8.1.1b) in terms of a map (*the constraint operator*)

$$\Phi : \mathscr{M}_2 \times \mathscr{S}^2 \to \mathscr{C} \times \mathscr{X}$$

such that for $h_{ij} \in \mathscr{M}_2$, $K_{ij} \in \mathscr{S}^2$ one has

$$\Phi \begin{pmatrix} h_{ij} \\ K_{ij} \end{pmatrix} \equiv \begin{pmatrix} r + K^2 - K_{ij}K^{ij} \\ -D^j K_{ij} + D_i K \end{pmatrix}.$$

In terms of the latter, the constraints (8.1.1a) and (8.1.1b) take the form

$$\Phi \begin{pmatrix} h_{ij} \\ K_{ij} \end{pmatrix} = 0.$$

The *linearisation of the constraint operator* Φ, $D\Phi : \mathscr{S}^2 \times \mathscr{S}^2 \to \mathscr{C} \times \mathscr{X}$, evaluated at (h_{ij}, K_{ij}) can be found to be given by

$$D\Phi \begin{pmatrix} \gamma_{ij} \\ Q_{ij} \end{pmatrix} = \begin{pmatrix} D^i D^j \gamma_{ij} - r_{ij}\gamma^{ij} - \Delta_{\boldsymbol{h}}\gamma + H \\ -D^j Q_{ij} + D_i Q - F_i \end{pmatrix}$$

where $\gamma \equiv h^{ij}\gamma_{ij}$, $Q \equiv h^{ij}Q_{ij}$ and

$$\begin{aligned} H &\equiv 2(KQ - K^{ij}Q_{ij}) + 2(K^{ki}K^j{}_k - KK^{ij})\gamma_{ij}, \\ F_i &\equiv \left(D_i K^{kj} - D^k K^j{}_i\right)\gamma_{jk} - \left(K^k{}_i D^j - \tfrac{1}{2}K^{kj}D_i\right)\gamma_{jk} + \tfrac{1}{2}K^k{}_i D_k \gamma, \end{aligned}$$

while $\Delta_{\boldsymbol{h}} \equiv h^{ij}D_iD_j$ is the Laplacian of the metric h_{ij}. Moreover, using integration by parts, the formal adjoint of the linearised constraint operator, $D\Phi^* : \mathscr{C} \times \mathscr{X} \to \mathscr{S}^2 \times \mathscr{S}^2$, can be seen to be given by

$$D\Phi^* \begin{pmatrix} X \\ X_i \end{pmatrix} = \begin{pmatrix} D_iD_jX - Xr_{ij} - \Delta_h X h_{ij} + H_{ij} \\ D_{(i}X_{j)} - D^k X_k h_{ij} + F_{ij} \end{pmatrix}$$

where

$$\begin{aligned} H_{ij} \equiv{}& 2X(K^k{}_i K_{jk} - KK_{ij}) - K_{k(i}D_{j)}X^k + \tfrac{1}{2}K_{ij}D_kX^k \\ &+ \tfrac{1}{2}K_{kl}D^kX^l h_{ij} - \tfrac{1}{2}X^k D_k K_{ij} + \tfrac{1}{2}X^k D_k K h_{ij}, \\ F_{ij} \equiv{}& 2X(Kh_{ij} - K_{ij}). \end{aligned}$$

Note that in the case of time-symmetric data, $H = F_i = H_{ij} = F_{ij} \equiv 0$, and the above expressions for $D\Phi$ and $D\Phi^*$ thereby reduce to those given in [11].

Remark 56. A calculation shows that $D\Phi^* = 0$ is equivalent to the KID equations (8.1.2a)-(8.1.2b). Indeed, one has that

$$D\Phi^* \begin{pmatrix} N \\ -2Y_i \end{pmatrix} = \begin{pmatrix} M_{ij} - M_k{}^k h_{ij} - \tfrac{1}{2}K_{kl}L^{kl}h_{ij} + \tfrac{1}{2}K_{ij}L_k{}^k \\ L_{ij} - L_k{}^k h_{ij} \end{pmatrix}$$

from which we see that $D\Phi^*(N,-2Y_i)=0$ if and only if $L_{ij}=M_{ij}=0$ —i.e. if and only if (N,Y^i) satisfy the KID equations.

Now, let $\mathscr{S}_{1,2}$ denote the space of covariant rank-3 tensors which are symmetric in the last two indices. Following Dain [11], we consider an operator $\mathcal{P}:\mathscr{S}^2\times\mathscr{S}_{1,2}\rightarrow\mathscr{C}\times\mathscr{X}$ such that

$$\mathcal{P}\begin{pmatrix}\gamma_{ij}\\ q_{kij}\end{pmatrix}\equiv D\Phi\begin{pmatrix}\gamma_{ij}\\ -D^k q_{kij}\end{pmatrix}$$

with formal adjoint, $\mathcal{P}^*:\mathscr{C}\times\mathscr{X}\rightarrow\mathscr{S}^2\times\mathscr{S}_{1,2}$, given by

$$\mathcal{P}^*\begin{pmatrix}X\\ X_i\end{pmatrix}\equiv\begin{pmatrix}1&0\\ 0&D_k\end{pmatrix}\cdot D\Phi^*\begin{pmatrix}X\\ X_i\end{pmatrix}=\begin{pmatrix}D_iD_jX-Xr_{ij}-\Delta_hXh_{ij}+H_{ij}\\ D_k(D_{(i}X_{j)}-D^lX_lh_{ij}+F_{ij})\end{pmatrix}.$$

Further, we consider the composition $\mathcal{P}\circ\mathcal{P}^*:\mathscr{C}\times\mathscr{X}\rightarrow\mathscr{C}\times\mathscr{X}$, given by

$$\mathcal{P}\circ\mathcal{P}^*\begin{pmatrix}X\\ X_i\end{pmatrix}\equiv\begin{pmatrix}2\Delta_{\boldsymbol{h}}\Delta_{\boldsymbol{h}}X-r^{ij}D_iD_jX+2r\Delta_{\boldsymbol{h}}X+\frac{3}{2}D^irD_iX+(\frac{1}{2}\Delta_{\boldsymbol{h}}r+r_{ij}r^{ij})X\\ +D^iD^jH_{ij}-\Delta_{\boldsymbol{h}}H_k{}^k-r^{ij}H_{ij}+\bar{H}\\ \\ D^j\Delta_{\boldsymbol{h}}D_{(i}X_{j)}+D_i\Delta_{\boldsymbol{h}}D^kX_k+D^j\Delta_{\boldsymbol{h}}F_{ij}-D_i\Delta_{\boldsymbol{h}}F_k{}^k-\bar{F}_i\end{pmatrix}$$

where

$$\begin{aligned}
&\bar{H}\equiv 2(K\bar{Q}-K^{ij}\bar{Q}_{ij})+2(K^{ki}K^j{}_k-KK^{ij})\bar{\gamma}_{ij},\\
&\bar{F}_i\equiv\left(D_iK^{kj}-D^kK^j{}_i\right)\bar{\gamma}_{jk}-\left(K^k{}_iD^j-\tfrac{1}{2}K^{kj}D_i\right)\bar{\gamma}_{jk}+\tfrac{1}{2}K^k{}_iD_k\bar{\gamma}\\
&\bar{\gamma}_{ij}\equiv D_iD_jX-Xr_{ij}-\Delta_{\boldsymbol{h}}Xh_{ij}+H_{ij}\\
&\bar{Q}_{ij}\equiv-\Delta_{\boldsymbol{h}}(D_{(i}X_{j)}-D^kX_kh_{ij}+F_{ij})
\end{aligned}$$

and $F_{ij},\ H_{ij}$ as above. One has the following:

Lemma 17. The operator $\mathcal{P}\circ\mathcal{P}^*:\mathscr{C}\times\mathscr{X}\rightarrow\mathscr{C}\times\mathscr{X}$ as defined above is a self-adjoint fourth order elliptic operator.

Proof. The self-adjointness follows from the definition as the operator is obtained by the composition of an operator and its formal adjoint. To verify the ellipticity of the operator we notice that the symbol is given by

$$\sigma_\xi\begin{pmatrix}X\\ X_i\end{pmatrix}=\begin{pmatrix}2|\xi|^2X\\ \xi^j|\xi|^2\xi_{(i}X_{j)}+\xi_i|\xi|^2\xi_jX^j\end{pmatrix}$$

for ξ_i a covector and $|\xi|^2\equiv\delta_{ij}\xi^i\xi^j$. Clearly, the first component is an isomorphism if $|\xi|^2\neq 0$. For the second component, contract first with ξ^i to get $2|\xi|^4\xi^jX_j=0$ for X_i in the kernel, which implies $\xi^jX_j=0$. Substituting back into the symbol, one obtains that $|\xi|^4X_i=0$. So, for $|\xi|^2\neq 0$, the symbol is injective. Clearly the codomain has the same dimension as the domain, and therefore σ_ξ is an isomorphism for $|\xi|^2\neq 0$ —i.e. $\mathcal{P}\circ\mathcal{P}^*$ is fourth-order elliptic operator. □

The previous discussion suggests the following:

Definition 14. The equation

$$\mathcal{P}\circ\mathcal{P}^*\begin{pmatrix}X\\ X_i\end{pmatrix}=0 \tag{8.1.3}$$

will be called the approximate Killing initial data (KID) equation and a solution (X, X^i) thereof an approximate Killing initial data set —or approximate KID for brevity.

Remark 57. As pointed out in [11], the equation $\mathcal{P}\circ\mathcal{P}^*(X, X_i) = 0$ is the Euler–Lagrange equation of the action

$$\int_{\mathcal{U}} \mathcal{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} \cdot \mathcal{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} \mathrm{d}\mu$$

Note that, had we used the operator $D\Phi^*$ rather than $\mathcal{P}^*$, then the pointwise norm defined by the integrand would contain terms of inconsistent physical dimension: $[X] = L^{-2},\ [X_i] = 1$, and so for instance $\big[(D_iD_jX)(D^iD^jX)\big] = L^{-6}$, while $\big[D_{(i}X_{j)}D^{(i}X^{j)}\big] = L^{-4}$.

8.1.3 Integration by parts identities

The expressions in the previous subsection and several of our arguments in latter parts are based on integration by parts. For quick reference, in this subsection we provide the integral expressions relating the operators $\mathcal{P}$ and $\mathcal{P}^*$ including boundary terms.

Let $\mathcal{U} \subset \mathcal{S}$ denote a compact set with boundary $\partial\mathcal{U}$. Recall that by definition

$$\begin{aligned}
\int_{\mathcal{U}} \begin{pmatrix} X \\ X^i \end{pmatrix} \cdot \mathcal{P} \begin{pmatrix} \gamma_{ij} \\ q_{kij} \end{pmatrix} \mathrm{d}\mu &= \int_{\mathcal{U}} \begin{pmatrix} X \\ X^i \end{pmatrix} \cdot D\Phi \begin{pmatrix} \gamma_{ij} \\ -D^k q_{kij} \end{pmatrix} \mathrm{d}\mu \\
&= \int_{\mathcal{U}} \begin{pmatrix} X \\ X^i \end{pmatrix} \cdot \begin{pmatrix} D^iD^j\gamma_{ij} - r_{ij}\gamma^{ij} - \Delta_{\boldsymbol{h}}\gamma + H \\ D^jD^k q_{kij} - D_iD^k q_{kj}{}^j - F_i \end{pmatrix} \mathrm{d}\mu \\
&= \int_{\mathcal{U}} X \left(D^iD^j\gamma_{ij} - r_{ij}\gamma^{ij} - \Delta_{\boldsymbol{h}}\gamma + H \right) \mathrm{d}\mu \\
&\qquad + \int_{\mathcal{U}} X^i \left(D^jD^k q_{kij} - D_iD^k q_{kj}{}^j - F_i \right) \mathrm{d}\mu \\
&= J_1 + J_2.
\end{aligned}$$

We now proceed to use integration by parts on J_1 and J_2. A lengthy computation shows that

$$\begin{aligned}
J_1 &\equiv \int_{\mathcal{U}} X \left(D^iD^j\gamma_{ij} - r_{ij}\gamma^{ij} - \Delta_{\boldsymbol{h}}\gamma + H \right) \mathrm{d}\mu \\
&= \int_{\mathcal{U}} \gamma_{ij} \left(D^jD^iX - h^{ij}\Delta_{\boldsymbol{h}}X - Xr^{ij} + 2(K^{ki}K^j{}_k - KK^{ij}) \right) \mathrm{d}\mu \\
&\qquad + \int_{\mathcal{U}} 2q_{kij}(h^{ij}XD^kK + h^{ij}KD^kX - XD^kK^{ij} - K^{ij}D^kX) d\mu \\
&\qquad + \oint_{\partial\mathcal{U}} n^k(\mathcal{A}_k + \mathcal{B}_k) \mathrm{d}S
\end{aligned}$$

where the boundary integrands are given by

$$\begin{aligned}
&\mathcal{A}_k \equiv XD^j\gamma_{jk} - D^jX\gamma_{jk} - D_kX\gamma - XD_k\gamma, \\
&\mathcal{B}_k \equiv 2(K^{ij}q_{kij} - Kq_{kj}{}^j)X.
\end{aligned}$$

Similarly, one finds that

$$
\begin{aligned}
J_2 \equiv& \int_{\mathcal{U}} X^i \left(D^j D^k q_{kij} - D_i D^k q_{kj}{}^j - F_i\right) \mathrm{d}\mu \\
=& \int_{\mathcal{U}} q_{kij} \left(D^k D^j X^i + D^k D_l h^{ij}\right) \mathrm{d}\mu \\
&\quad - \int_{\mathcal{U}} \gamma_{jk} \left((D_i K^{kj} - D^k K^j{}_i) X^i + D^j (X^i K_i{}^k) - \tfrac{1}{2} D_i (X^i K^{kj}) - \tfrac{1}{2} h^{jk} D_i (X^l K_l{}^i)\right) \mathrm{d}\mu \\
&\quad + \oint_{\partial\mathcal{U}} n^k (\mathcal{C}_k + \mathcal{D}_k) \mathrm{d}S
\end{aligned}
$$

where the boundary integrands are given by

$$
\begin{aligned}
\mathcal{C}_k &\equiv X^i D^l q_{lik} - D^j X^i q_{kij} + D_i X^i q_{kj}{}^j - X_i D^l q_{lj}{}^j, \\
\mathcal{D}_k &\equiv X^i K_i{}^l \gamma_{kl} - \tfrac{1}{2} X_k K^{lj} \gamma_{jl} - \tfrac{1}{2} X^i K_{ik} \gamma.
\end{aligned}
$$

Putting everything together and after some further manipulations one finds the identity

$$
\int_{\mathcal{U}} \begin{pmatrix} X \\ X^i \end{pmatrix} \cdot \mathcal{P} \begin{pmatrix} \gamma_{ij} \\ q_{kij} \end{pmatrix} \mathrm{d}\mu = \int_{\mathcal{U}} \begin{pmatrix} \gamma^{ij} \\ q^{kij} \end{pmatrix} \cdot \mathcal{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} + \oint_{\partial\mathcal{U}} n^k \big(\mathcal{A}_k + \mathcal{B}_k + \mathcal{C}_k + \mathcal{D}_k\big) \mathrm{d}S. \quad (8.1.4)
$$

8.2 The approximate Killing vector equation on asymptotically Euclidean manifolds

In this section we study the solvability of the approximate KID equation on asymptotically Euclidean manifolds. The standard methods to study elliptic equations on this type of manifolds employ so-called *weighted Sobolev spaces* —thus, we start by briefly reviewing our basic technical tools in Section 8.2.1. The key assumption on the class of initial data sets to be considered are discussed in 8.2.2. The existence results for the approximate KID equation are given in Subsection 8.1.3.

8.2.1 Weighted Sobolev spaces

In order to discuss the decay of the various tensor fields in the 3-manifold $\mathcal{S}$ we need to make use of *weighted Sobolev spaces* —see e.g. [52, 90–92]. Given an arbitrary point $p \in \mathcal{S}$ one defines for $x \in \mathcal{S}$

$$
\sigma(x) \equiv \left(1 + d(p,x)^2\right)^{1/2}
$$

where $d(p,x)$ denotes the Riemannian distance on $\mathcal{S}$. The function σ is used to define the weighted L^2-norm

$$
\parallel u \parallel_\delta \equiv \left(\int_{\mathcal{S}} |u|^2 \sigma^{-2\delta-3} \mathrm{d}^3 x\right)^{1/2}, \qquad \delta \in \mathbb{R}.
$$

In particular, if $\delta = -3/2$ one recovers the usual L^2-norm. Different choices of origin give rise to equivalent weighted norms.

Remark 58. In the above and in the rest of the Chapter, we follow Bartnik's conventions [92] to denote the weighted Sobolev spaces and norms.

The fall-off behaviour of the various fields will be expressed in terms of weighted Sobolev spaces

H^s_δ consisting of functions for which

$$\| u \|_{s,\delta} \equiv \sum_{0\leq|\alpha|\leq s} \| D^\alpha u \|_{\delta-|\alpha|} < \infty,$$

where s is a non-negative integer, and where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. One says that $u \in H^\infty_\delta$ if $u \in H^s_\delta$ for all s. We will say that a tensor belongs to a given function space if its norm does.

In the following given some coordinates $x = (x^\alpha)$, let $|x|^2 \equiv \delta_{\alpha\beta}x^\alpha x^\beta$. We will make repeated use of the following result[1]:

Lemma 18. Let $u \in H^\infty_\delta$. Then u is smooth (i.e. C^∞) over $\mathcal{S}$ and has a fall-off at infinity such that

$$D^l u = o\big(|x|^{\delta-|l|}\big).$$

The proof can be found in [92] —see also Section 6.1 in [86]. The following *Multiplication Lemma* has been proven in [86]:

Lemma 19. Let $u = o_\infty(|x|^{\delta_1})$, $v = o_\infty(|x|^{\delta_2})$ and $w = O(|x|^\gamma)$. Then

$$uv = o_\infty(|x|^{\delta_1+\delta_2}), \qquad uw = o_\infty(|x|^{\delta_1+\gamma}).$$

This lemma can be readily extended to tensor fields.

8.2.2 Decay assumptions

In what follows we will consider initial data sets $(\mathcal{S}, h_{ij}, K_{ij})$ for the vacuum Einstein field equations possessing, in principle, several *asymptotically Euclidean ends*. Thus, we assume there exists a compact set $\mathcal{B}$ such that

$$\mathcal{S} \setminus \mathcal{B} = \sum_{k=1}^{n} \mathcal{S}_{(k)}$$

where $\mathcal{S}_{(k)}$, $k = 1, \ldots, n$, are open sets diffeomorphic to the complement of a closed ball on $\mathbb{R}^3$. Each set $\mathcal{S}_{(k)}$ is called an *asymptotic end*. On each of these ends one can introduce (non-unique) *asymptotically Cartesian* coordinates $x = (x^\alpha)$. Our basic decay assumptions for the fields h_{ij} and K_{ij} are expressed in terms of these coordinates:

Assumption 1 (Decay Assumptions)**.** *On each asymptotically Euclidean end one has*

$$\begin{aligned} h_{\alpha\beta} - \delta_{\alpha\beta} &= o_\infty(|x|^{-1/2}), \\ K_{\alpha\beta} &= o_\infty(|x|^{-3/2}). \end{aligned}$$

The following definition will prove useful:

Definition 15. An asymptotically Euclidean initial data set $(\mathcal{S}, h_{ij}, K_{ij})$ satisfying the Decay Assumptions 1 is said to be *stationary* if there exists non-trivial $(N, N^i) \in H^2_{1/2}$ such that

$$\mathcal{P}^* \begin{pmatrix} N \\ N^i \end{pmatrix} = 0. \tag{8.2.1}$$

[1] Recall that $f(x) = o(|x|^\alpha)$ if $f(x)/|x|^\alpha \to 0$ as $|x| \to \infty$. If $\partial^n f(x) = o(|x|^{\alpha-n})$ for each non-negative integer, then we write $f(x) = o_\infty(|x|^\alpha)$.

Remark 59. As it is to be expected, a stationary initial data set (in the sense of Definition 15) admits a KID. To see this, observe that equation (8.2.1) implies that

$$D_i D_j N - N r_{ij} - \Delta_{\boldsymbol{h}} N h_{ij} + H_{ij} = 0, \tag{8.2.2a}$$

$$D_k\big(D_{(i}N_{j)} - D^l N_l h_{ij} + F_{ij}\big) = 0. \tag{8.2.2b}$$

Direct inspection shows that

$$D_{(i}N_{j)} - D^l N_l h_{ij} + F_{ij} = o(|x|^{-1/2}),$$

It follows from (8.2.2b) that the above is a covariantly constant tensor field and therefore must vanish —i.e.

$$D_{(i}N_{j)} - D^l N_l h_{ij} + F_{ij} = 0.$$

Combining this observation with (8.2.2a), we see that $D\Phi^*(N, N^i) = 0$ and hence $(N, -\frac{1}{2}N^i)$ solves the KID equations (8.1.2a)-(8.1.2b) —see Remark 56. Finally, we observe that the behaviour $(N, -\frac{1}{2}N^i) = o(|x|^{1/2})$ for a KID is only consistent with *translational Killing vector fields* —i.e. Killing vectors which to leading order look like a (timelike, spatial or null) translation in the Minkowski spacetime. Now, the only type of translational Killing vector a spacetime with non-vanishing ADM 4-momentum can admit is one which is timelike and bounded at infinity —i.e. a stationary Killing vector, see Section III in [93]. Clearly the reverse is also true: if an initial data set admits a stationary Killing vector, then the data is stationary in the sense of Definition 15. It should be stressed that *our definition of stationary initial data sets excludes initial data sets for the Minkowski spacetime* as these necessarily have a vanishing ADM 4-momentum. The condition on the ADM 4-momentum in Definition 15 arises from the need to single out the stationary Killing vector field among out of the collection of translational Killing vectors.

The asymptotic behaviour of solutions to the KID equations has been studied in [93] from where we adapt the following result:

Proposition 25. Let $(\mathcal{S}, h_{ij}, K_{ij})$ denote a smooth vacuum initial data set satisfying the Decay Assumptions 1. Moreover, let N, Y^i be, respectively, a smooth scalar field and a vector field over $\mathcal{S}$ satisfying the KID equations. Then, there exists a constant tensor with components $\mathfrak{L}_{\mu\nu} = \mathfrak{L}_{[\mu\nu]}$ such that

$$N - \mathfrak{L}_{0\alpha}x^\alpha = o_\infty(|x|^{1/2}), \qquad Y^\alpha - \mathfrak{L}_{\alpha\beta}x^\beta = o_\infty(|x|^{1/2}).$$

If $\mathfrak{L}_{\mu\nu} = 0$, then there exists a constant vector with components $\mathfrak{A}^\mu$ such that

$$N - \mathfrak{A}^0 = o_\infty(|x|^{-1/2}), \qquad Y^\alpha - \mathfrak{A}^\alpha = o_\infty(|x|^{-1/2}).$$

Finally, if $\mathfrak{L}_{\mu\nu} = \mathfrak{A}^\mu = 0$, then $N = 0$ and $Y^i = 0$.

8.2.3 Basic results of the theory of elliptic equations on asymptotically Euclidean manifolds

In view of the Decay Assumptions 1, the approximate KID equation (8.1.3) can be written, in local coordinates, in the form

$$\mathcal{L}\mathbf{u} \equiv (\mathbf{A}^{\alpha\beta\gamma\delta} + \mathbf{a}^{\alpha\beta\gamma\delta}) \cdot \partial_\alpha\partial_\beta\partial_\gamma\partial_\delta\mathbf{u} + \mathbf{a}^{\alpha\beta\gamma} \cdot \partial_\alpha\partial_\beta\partial_\gamma\mathbf{u} + \mathbf{a}^{\alpha\beta} \cdot \partial_\alpha\partial_\beta\mathbf{u} + \mathbf{a}^\alpha \cdot \partial_\alpha\mathbf{u} + \mathbf{a} \cdot \mathbf{u} = 0,$$

where $\mathbf{u} : \mathcal{S} \to \mathbb{R}^4$ is a vector-valued function over $\mathcal{S}$, $\mathbf{A}^{\alpha\beta\gamma\delta}$ denote constant matrices, while $\mathbf{a}^{\alpha\beta\gamma\delta}$, $\mathbf{a}^{\alpha\beta\gamma}$, $\mathbf{a}^{\alpha\beta}$, $\mathbf{a}^{\alpha}$ and $\mathbf{a}$ denote smooth matrix-valued functions of the coordinates $x = (x^\alpha)$.

The operator $\mathcal{L}$ is said to be *asymptotically homogeneous* if

$$\mathbf{a}^{\alpha\beta\gamma\delta} \in H^\infty_\tau, \qquad \mathbf{a}^{\alpha\beta\gamma} \in H^\infty_{\tau-1}, \qquad \mathbf{a}^{\alpha\beta} \in H^\infty_{\tau-2}, \qquad \mathbf{a}^{\alpha} \in H^\infty_{\tau-3}, \qquad \mathbf{a} \in H^\infty_{\tau-4},$$

for some $\tau < 0$ —see e.g. [52, 90].

Remark 60. Direct inspection using the Decay Assumptions 1 imply that $\mathcal{L}$ is asymptotically homogeneous with $\tau = -1/2$. This is the standard assumption when working with weighted Sobolev spaces.

In the following we will make use of the following version of the *Fredholm alternative* for fourth-order asymptotically homogeneous operators on asymptotically Euclidean manifolds —see [52]:

Proposition 26. Let $\mathcal{L}$ be an asymptotically homogeneous elliptic operator of order 4 with smooth coefficients. Given δ not a negative integer, the equation

$$\mathcal{L}\mathbf{u} = \mathbf{f}, \qquad \mathbf{f} \in H^0_{\delta-4}$$

has a solution $\mathbf{u} \in H^4_\delta$ if and only if

$$\int_{\mathcal{S}} \mathbf{f} \cdot \mathbf{v}\, d\mu = 0$$

for all $\mathbf{v}$ satisfying

$$\mathcal{L}^*\mathbf{v} = 0, \qquad \mathbf{v} \in H^0_{1-\delta},$$

where $\mathcal{L}^*$ denotes the formal adjoint of $\mathcal{L}$.

Finally, to assert the regularity of solutions we need the following elliptic estimate —see Theorem 6.3. of [52]:

Proposition 27. Let $\mathcal{L}$ be an asymptotically homogeneous elliptic operator of order 4 with smooth coefficients. Then for any $\delta \in \mathbb{R}$ and any $s \geq 4$, there exists a constant C such that for every $\mathbf{v} \in H^s_{loc} \cap H^0_\delta$, the following inequality holds:

$$\parallel \mathbf{v} \parallel_{H^s_\delta} \leq C\big(\parallel \mathcal{L}\mathbf{v} \parallel_{H^{s-4}_{\delta-2}} + \parallel \mathbf{v} \parallel_{H^{s-4}_\delta} \big).$$

In the above proposition H^s_{loc} denotes the local Sobolev space —that is, $\mathbf{v} \in H^s_{loc}$ if for an arbitrary smooth function ϕ with compact support, $\phi\mathbf{v} \in H^s$.

Remark 61. If $\mathcal{L}$ has smooth coefficients and $\mathcal{L}\mathbf{v} = 0$, then it follows that all the H^s_δ norms of $\mathbf{v}$ are bounded by the H^0_δ norm. Thus, it follows that if a solution to $\mathcal{L}\mathbf{v} = 0$ exists, it must be smooth —*elliptic regularity.*

8.2.4 Existence of solutions to the approximate Killing vector equation

We are now in the position of analysing the existence of solutions to the approximate Killing equation (8.1.3). Our main tools will be the Fredholm alternative and integration by parts. We begin by considering some auxiliary results.

Auxiliary existence results

The following result relating solutions to the approximate Killing equations to solutions to the KID equations will be needed in our main result:

Lemma 20. Let $(\mathcal{S}, h_{ij}, K_{ij})$ be a complete, smooth asymptotically Euclidean initial data set for the Einstein vacuum field equations with n asymptotic ends satisfying the Decay Assumptions 1. Then, for $0 < \beta \leq 1/2$,

$$\ker\{\mathcal{P}\circ\mathcal{P}^* : H^\infty_\beta \to H^\infty_{\beta-4}\} = \ker\{\mathcal{P}^* : H^\infty_\beta \to H^\infty_{\beta-2}\}$$

That is to say, the equation

$$\mathcal{P}\circ\mathcal{P}^*\begin{pmatrix} N \\ N^i \end{pmatrix} = 0$$

admits a solution $(N, N^i) \in H^\infty_\beta$, $0 < \beta \leq 1/2$, if and only if $(\mathcal{S}, h_{ij}, K_{ij})$ is stationary in the sense of Definition 15. Moreover, if the solution exists then it is unique up to constant rescaling.

Proof. Assume that $\mathcal{P}\circ\mathcal{P}^*(N, N^i) = 0$. Making use of the identity (8.1.4) with

$$\begin{pmatrix} \gamma^{ij} \\ q^{kij} \end{pmatrix} = \mathcal{P}^*\begin{pmatrix} N \\ N^i \end{pmatrix}$$

one finds that

$$\int_{\mathcal{S}} \mathcal{P}^*\begin{pmatrix} N \\ N^i \end{pmatrix} \cdot \mathcal{P}^*\begin{pmatrix} N \\ N^i \end{pmatrix} \mathrm{d}\mu = -\oint_{\partial\mathcal{S}_\infty} n^k(\mathcal{A}_k + \mathcal{B}_k + \mathcal{C}_k + \mathcal{D}_k)\mathrm{d}S$$

where $\partial\mathcal{S}_\infty$ denotes the sphere at infinity. We proceed now to evaluate the various boundary terms.

We observe that under the Decay Assumptions 1 direct inspection shows that

$$H_{ij} = o(|x|^{-2}),$$

from where it follows that

$$\gamma_{ij} = D_i D_j N - N r_{ij} - \Delta N h_{ij} + H_{ij} = o(|x|^{-3/2}).$$

Hence, one has that

$$\mathcal{A}_k = N D^i \gamma_{ik} - D^i N \gamma_{ik} + D_k N \gamma - N D_k \gamma = o(|x|^{-2}).$$

Thus, taking into account that $\mathrm{d}S = O(|x|^2)$ one concludes that

$$\oint_{\partial\mathcal{S}_\infty} n^k \mathcal{A}_k \mathrm{d}S = 0.$$

Next, we consider

$$\mathcal{C}_k = N^i D^l q_{lik} - D^j N^i q_{kij} + D_i N^i q_{kj}{}^j - N_i D^l q_{lj}{}^j$$

where

$$q_{kij} = D_k\big(D_{(i}N_{j)} - D^l N_l h_{ij} - F_{ij}\big), \qquad F_{ij} = 2N(K h_{ij} - K_{ij}).$$

From the Decay Assumptions 1 it follows that in this case

$$F_{ij} = o(|x|^{-1}), \qquad q_{kij} = o(|x|^{-3/2})$$

so that

$$\mathcal{C}_k = o(|x|^{-2}).$$

Thus, one has that

$$\oint_{\partial\mathcal{S}_\infty} \mathcal{C}_k n^k \mathrm{d}S = 0.$$

Finally, similar considerations give that

$$\mathcal{D}_k = \frac{1}{2} N_k K^{lj}\gamma_{jl} + \frac{1}{2} N^i K_{ik}\gamma - N^i K_i{}^l \gamma_{kl} = o(|x|^{-5/2})$$

so that

$$\oint_{\partial\mathcal{S}_\infty} n^k \mathcal{D}_k \mathrm{d}S = 0.$$

From the previous discussion it follows then that

$$\int_{\mathcal{S}} \mathcal{P}^* \begin{pmatrix} N \\ N^i \end{pmatrix} \cdot \mathcal{P}^* \begin{pmatrix} N \\ N^i \end{pmatrix} \mathrm{d}\mu = 0$$

so that $\mathcal{P}^*(N, N^i) = 0$, and therefore that the data is stationary. Finally, uniqueness of the solution follows from Proposition 25. Suppose, for contradiction, that there exist two distinct solutions, giving rise to two distinct KID sets $(N, -\frac{1}{2}N^i)$ and $(\tilde{N}, -\frac{1}{2}\tilde{N}^i)$. Taking the appropriate linear combination we arrive at a KID set with a lapse that goes to zero at infinity while the shift is in H^∞_β, $\beta \leq 1/2$ —that is, one has a KID associated to a spatial translation. This contradicts the fact that the ADM 4-momentum of the initial data is non-vanishing —see Section III in [93]. □

Remark 62. Making use of the asymptotic expansion provided by Proposition 25 one finds that for stationary initial data sets, the solutions provided by Lemma 20 are of the form:

$$N - \mathfrak{A}^0 = o_\infty(|x|^{-1/2}), \qquad N^\alpha - \mathfrak{A}^\alpha = o_\infty(|x|^{-1/2}) \tag{8.2.3}$$

with the components of a $\mathfrak{A}^\mu$ a constant vector field.

Main existence result

Following [11] we now will look for solutions of the approximate Killing equation such that

$$\begin{aligned} &X = \mathfrak{D}|x| + \vartheta, \qquad \vartheta \in H^\infty_{1/2}, \\ &X^i \in H^\infty_{1/2}. \end{aligned}$$

in each asymptotically Euclidean end and where $\mathfrak{D}$ is a constant. This ansatz is motivated by the observation that $\Delta^2_{\boldsymbol{\delta}}|x| = 0$, with $\Delta_{\boldsymbol{\delta}}$ the flat Laplacian —that is, the blowing up term $\mathfrak{D}|x|$ is in the kernel of the first component of the operator $\mathcal{P} \circ \mathcal{P}^*$ evaluated on the 3-dimensional flat metric.

Theorem 8. Let $(\mathcal{S}, h_{ij}, K_{ij})$ be a complete, smooth asymptotically Euclidean initial data set for the Einstein vacuum field equations with n asymptotic ends. Then there exists a solution (X, X^i)

to the approximate KID equation,

$$\mathcal{P}\circ\mathcal{P}^*\begin{pmatrix} X \\ X^i \end{pmatrix} = 0,$$

such that at each asymptotic end one has the asymptotic behaviour

$$\begin{aligned} &X_{(k)} = \mathfrak{D}_{(k)}|x| + \vartheta_{(k)}, \qquad \vartheta_{(k)} \in H^\infty_{1/2}, \\ &X^i_{(k)} \in H^\infty_{1/2}, \end{aligned}$$

where $\mathfrak{D}_{(k)}$, $k = 1,\ldots,n$, are constants and $\mathfrak{D}_{(k)} = 0$ for some k if and only if $(\mathcal{S}, h_{ij}, K_{ij})$ is stationary in the sense of Definition 15.

Proof. Substituting the above ansatz in equation (8.1.3) one obtains

$$\mathcal{P}\circ\mathcal{P}^*\begin{pmatrix} \vartheta \\ X^i \end{pmatrix} = -\mathcal{P}\circ\mathcal{P}^*\begin{pmatrix} \mathfrak{D}|x| \\ 0 \end{pmatrix}. \tag{8.2.4}$$

Under the Decay Assumptions 1, a lengthy computation shows that

$$\begin{gathered} H_{ij} = o(|x|^{-2}), \qquad F_{ij} = o(|x|^{-1/2}), \qquad Q_{ij} = o(|x|^{-5/2}), \qquad \bar{\gamma}_{ij} = o(|x|^{-1}), \\ F_i = o(|x|^{-7/2}), \qquad H = o(|x|^{-4}), \end{gathered}$$

where, in particular, it has been used that

$$\partial_\alpha|x| = \frac{x_\alpha}{|x|} = O(1), \qquad \partial_\alpha\partial_\beta|x| = \frac{\delta_{\alpha\beta}}{|x|} - \frac{x_\alpha x_\beta}{|x|^3} = O(|x|^{-1}).$$

Hence,

$$\begin{aligned} &2\Delta_{\boldsymbol{h}}\Delta_{\boldsymbol{h}}X - r^{ij}D_iD_jX + 2r\Delta_{\boldsymbol{h}}X + D^iD^jH_{ij} - \Delta_{\boldsymbol{h}}H_k{}^k + \bar{H} \\ &\qquad\qquad +\tfrac{3}{2}D^irD_iX + (\tfrac{1}{2}\Delta_{\boldsymbol{h}}r + r_{ij}r^{ij})X - r^{ij}H_{ij} = o(|x|^{-7/2}), \\ &D^j\Delta_{\boldsymbol{h}}F_{ij} - D_i\Delta_{\boldsymbol{h}}F_k{}^k - \bar{F}_i = o(|x|^{-7/2}). \end{aligned}$$

so that

$$\mathcal{P}\circ\mathcal{P}^*\begin{pmatrix} \mathfrak{D}|x| \\ 0 \end{pmatrix} \in H^0_{-7/2}.$$

To prove the existence of solutions to equation (8.2.4) we make use of the Fredholm alternative in weighted Sobolev spaces, according to which equation (8.2.4) will have solution (ϑ, X^i) if and only if its right-hand-side is L^2-orthogonal to $\operatorname{coker}\{\mathcal{P}\circ\mathcal{P}^* : H^4_{1/2} \to H^0_{1/2}\}$ —i.e. if and only if

$$\int_{\mathcal{S}} \mathcal{P}\circ\mathcal{P}^*\begin{pmatrix} \mathfrak{D}|x| \\ 0 \end{pmatrix}\cdot\begin{pmatrix} N \\ N^i \end{pmatrix} \mathrm{d}\mu = 0$$

for all $(N, N^i) \in H^0_{1/2}$ for which

$$\mathcal{P}\circ\mathcal{P}^*\begin{pmatrix} N \\ N^i \end{pmatrix} = 0.$$

From Lemma 20 we know that this equation has non-trivial solutions (i.e. that $\operatorname{coker}\{\mathcal{P}\circ\mathcal{P}^* : H^4_{1/2} \to H^0_{1/2}\}$ will be non-trivial) if and only if $(\mathcal{S}, h_{ij}, K_{ij})$ is stationary. Thus, if the initial data set is not stationary, the Fredholm alternative guarantees a solution (ϑ, X^i) to (8.2.4).

For the stationary case, the cokernel is spanned by a single Killing vector with components (N, N_i), taking the form of (8.2.3). Let

$$\begin{pmatrix} \Gamma_{ij} \\ Q_{kij} \end{pmatrix} \equiv \mathcal{P}^* \begin{pmatrix} \mathfrak{D}|x| \\ 0 \end{pmatrix} = \begin{pmatrix} \mathfrak{D}(D_iD_j|x| - |x|r_{ij} - \Delta|x|h_{ij}) + H_{ij} \\ D_kF_{ij} \end{pmatrix}$$

where, now,

$$\begin{aligned} H_{ij} &\equiv 2\mathfrak{D}|x|(K^k{}_iK_{jk} - KK_{ij}) = o(|x|^{-2}), \\ F_{ij} &\equiv 2\mathfrak{D}|x|(Kh_{ij} - K_{ij}) = o(|x|^{-1/2}). \end{aligned}$$

It then follows that

$$\Gamma_{ij} = o(|x|^{-1}), \qquad Q_{kij} = o(|x|^{-3/2})$$

and that

$$\mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} \mathfrak{D}|x| \\ 0 \end{pmatrix} = \mathcal{P} \begin{pmatrix} \Gamma_{ij} \\ Q_{kij} \end{pmatrix} = o(|x|^{-7/2})$$

Then, using the identity (8.1.4) and the fact that, by assumption, $\mathcal{P}^*(N, N^i) = 0$, we see that

$$\int_{\mathcal{S}} \mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} \mathfrak{D}|x| \\ 0 \end{pmatrix} \cdot \begin{pmatrix} N \\ N_i \end{pmatrix} d\mu = \oint_{\partial\mathcal{S}_\infty} n^k(\mathcal{A}_k + \mathcal{B}_k + \mathcal{C}_k + \mathcal{D}_k)dS \tag{8.2.5}$$

where, here

$$\begin{aligned} \mathcal{A}_k &\equiv ND^j\Gamma_{jk} - D^jN\Gamma_{jk} - D_kN\Gamma - ND_k\Gamma, \\ \mathcal{B}_k &\equiv 2(K^{ij}Q_{kij} - KQ_{kj}{}^j)N, \\ \mathcal{C}_k &\equiv N^iD^lQ_{lik} - D^jN^iQ_{kij} + D_iN^iQ_{kj}{}^j - N_iD^lQ_{lj}{}^j, \\ \mathcal{D}_k &\equiv N^iK_i{}^l\Gamma_{kl} - \tfrac{1}{2}N_kK^{lj}\Gamma_{jl} - \tfrac{1}{2}N^iK_{ik}\Gamma, \end{aligned}$$

and $\Gamma \equiv h^{ij}\Gamma_{ij}$. We find then that

$$\mathcal{B}_k = o(|x|^{-3}), \qquad \mathcal{C}_k = o(|x|^{-5/2}), \qquad \mathcal{D}_k = o(|x|^{-5/2})$$

and

$$\mathcal{A}_k = -4\mathfrak{D}\mathfrak{A}^0|x|^{-2}n_k + o(|x|^{-5/2})$$

Therefore, the only contribution to the right-hand-side of (8.2.5) is the following

$$\oint_{\partial\mathcal{S}_\infty} n^k\mathcal{A}_kdS = -4\mathfrak{D}\mathfrak{A}^0 \oint_{\partial\mathcal{S}_\infty} |x|^{-2}dS = -16\pi\mathfrak{D}\mathfrak{A}^0$$

Since $\mathfrak{A}^0 \neq 0$, we see that in the stationary case we have an obstruction to solving (8.2.4), unless $\mathfrak{D} = 0$, in which case $(\vartheta, X_i) = (N, N_i)$ is the unique solution. □

Remark 63. The fact that $(\vartheta, N^i) \in H^\infty_{1/2}$ in the previous theorem follows from an application of Proposition 27 to equation (8.2.4).

8.3 The Dain invariant on conformally flat initial data sets

We have seen in the previous section that an approximate Killing vector with lapse of the form $\eta = \mathfrak{D}|x|+\vartheta$ exists for general asymptotically flat data, and moreover, that the constant $\mathfrak{D}$ vanishes if and only if the spacetime development is stationary. In this section we analyse further the asymptotic properties of the solutions to the approximate Killing vector equation in the case of conformally flat initial data sets.

8.3.1 Solutions to Poisson's equation in $\mathbb{R}^3$

We start with some mathematical preliminaries. Let us assume for the remainder of this section that $K_{ij} = O(|x|^{-3+\epsilon})$, for any $\epsilon > 0$. It follows then from the Hamiltonian constraint that

$$r = -K^2 + K_{ij}K^{ij} = O(|x|^{-6+2\epsilon}).$$

Moreover, the lapse component of the approximate Killing vector equation can be found to satisfy

$$2\Delta_{\boldsymbol{h}}\Delta_{\boldsymbol{h}}\eta - r^{ij}D_iD_j\eta + r_{ij}r^{ij}\eta = O(|x|^{-11/2+\epsilon}). \tag{8.3.1}$$

As is well known, the harmonic functions on $\mathbb{R}^3$ are spanned by functions of the forms

$$Q_{\alpha_1\cdots\alpha_k}x^{\alpha_1}\cdots x^{\alpha_k}, \qquad \frac{Q_{\alpha_1\cdots\alpha_k}x^{\alpha_1}\cdots x^{\alpha_k}}{|x|^{2k+1}}, \qquad k = 0,1,2,\ldots,$$

where $Q_{\alpha_1\cdots\alpha_k}$ are symmetric trace-free tensors with constant coefficients. The following result of Myers, [94], will prove useful:

Lemma 21. Let δ denote the flat 3–metric and $G = O(|x|^{-2-p-\epsilon}(\ln|x|)^q)$ a Hölder continuous function. Then the equation

$$\Delta_\delta V = G \tag{8.3.2}$$

admits a solution $V^\star$ satisfying

$$V^\star(x) = \begin{cases} O(|x|^{-p-\epsilon}(\ln|x|)^q) & \text{if } \; 0 < p < 1 \;\text{ or }\; \epsilon > 0, \\ O(|x|^{-p}(\ln|x|)^{q+1}) & \text{otherwise.} \end{cases}$$

Remark 64. By linearity of the Poisson equation (8.2.4), any two solutions thereof differ only by harmonic terms. In particular, the most general solution $V(x)$ of (8.3.2), assuming $V = O(|x|^r)$ for $r > -p$, is given by

$$V(x) = \begin{cases} V^\star(x) + \sum_{k=\lfloor -r\rfloor}^{\lfloor p-1\rfloor} Q_{\alpha_1\cdots\alpha_k}\dfrac{x^{\alpha_1}\cdots x^{\alpha_k}}{|x|^{2k+1}} & \text{if} \quad r < 0, \\ V^\star(x) + \sum_{k=0}^{\lfloor p-1\rfloor} Q_{\alpha_1\cdots\alpha_k}\dfrac{x^{\alpha_1}\cdots x^{\alpha_k}}{|x|^{2k+1}} + \displaystyle\sum_{l=0}^{\lfloor r\rfloor}\widehat{Q}_{\alpha_1\cdots\alpha_l}x^{\alpha_1}\cdots x^{\alpha_l} & \text{if} \quad r > 0. \end{cases}$$

for some symmetric, trace-free $Q_{\alpha_1\cdots\alpha_k}$, $\widehat{Q}_{\alpha_1\cdots\alpha_l}$ with constant coefficients and where for a real number p, $\lfloor p\rfloor$ denotes the floor of p —i.e. the largest integer smaller than p. It will be useful to note that, for $k \in \mathbb{Z}$,

$$\Delta_{\boldsymbol{\delta}} \left(\frac{x^\alpha}{|x|^k} \right) = k(k-3) \frac{x^\alpha}{|x|^{k+2}}.$$

8.3.2 Conformally flat initial data sets

We consider now maximal conformally-flat data initial data sets, i.e. collections $(\mathcal{S}, h_{ij}, K_{ij})$ such that

$$h_{ij} = \phi^4 \delta_{ij}, \qquad K_{ij} = P_{ij}$$

where $\phi \to 1$ as $|x| \to \infty$ and $P_{\alpha\beta} = O(|x|^{-3+\epsilon})$ is a symmetric, tracefree and divergence free with respect to the flat metric. It will also prove convenient to define $\psi_{\alpha\beta} \equiv \phi^2 P_{\alpha\beta}$, in terms of which the Hamiltonian and momentum constraints take the familiar forms

$$\Delta_{\boldsymbol{\delta}} \phi = -\tfrac{1}{8} \phi^{-7} \psi_{\alpha\beta} \psi^{\alpha\beta}, \tag{8.3.3a}$$

$$\partial^\alpha \psi_{\alpha\beta} = 0, \tag{8.3.3b}$$

where indices are now raised and lowered with respect to the flat metric, δ_{ij}. Then, it follows from (8.3.3a) and an application of Lemma 21 that

$$\phi = 1 + \frac{2m}{|x|} + \frac{L_\alpha x^\alpha}{|x|^3} + \frac{A_{\alpha\beta} x^\alpha x^\beta}{|x|^5} + O\left(\frac{\ln|x|}{|x|^{4-2\epsilon}} \right) \tag{8.3.4}$$

for some constant m, and constant-coefficient L_α, $A_{\alpha\beta}$, which are independent of the extrinsic curvature $P_{\alpha\beta}$ which contributes only at order $O(\ln|x|/|x|^{4-2\epsilon})$.

In terms of the flat connection, equation (8.3.1) becomes

$$\begin{aligned} \Delta_{\boldsymbol{\delta}} \Delta_{\boldsymbol{\delta}} \eta + A(\phi)^\alpha \partial_\alpha \Delta_{\boldsymbol{\delta}} \eta + B(\phi)^{\alpha\beta} \partial_\alpha \partial_\beta \eta + B(\phi) \Delta_{\boldsymbol{\delta}} \eta& \\ + C(\phi)^\alpha \partial_\alpha \eta + D(\phi) \eta = O(|x|^{-11/2+\epsilon})& \end{aligned} \tag{8.3.5}$$

where

$$\begin{aligned}
A(\phi)^\alpha &\equiv -4\phi^{-1} \partial^\alpha \phi, \\
B(\phi)^{\alpha\beta} &\equiv \phi^{-2} (5\phi \partial^\alpha \partial^\beta \phi - 19 \partial^\alpha \phi \partial^\beta \phi), \\
B(\phi) &\equiv 13\phi^{-2} |\partial\phi|^2, \\
C(\phi)^\alpha &\equiv 4\phi^{-3} (12|\partial\phi|^2 \partial^\alpha \phi - 5\phi (\partial_\beta \phi) \partial^\beta \partial^\alpha \phi), \\
D(\phi) &\equiv 2\phi^{-4} (6|\partial\phi|^4 - 6\phi (\partial^\alpha \phi)(\partial^\beta \phi) \partial_\alpha \partial_\beta \phi + \phi^2 (\partial^\alpha \partial^\beta \phi)(\partial_\alpha \partial_\beta \phi)).
\end{aligned}$$

Proposition 28. Let $(\mathcal{S}, h_{ij}, P_{ij})$ be maximal conformally-flat data with $P_{\alpha\beta} = O(|x|^{-3+\epsilon})$, then the lapse of the approximate Killing vector has an asymptotic expansion of the form

$$\begin{aligned} \eta = \mathfrak{D}|x| + 18\mathfrak{D} m \ln|x| + Q_\alpha \frac{x^\alpha}{|x|} + Q^{(1)} - 104\mathfrak{D} m^2 \frac{\ln|x|}{|x|}& \\ + \frac{Q^{(2)}}{|x|} - \frac{1}{4}(23\mathfrak{D} L_\alpha - 26 m Q_\alpha) \frac{x^\alpha}{|x|^2} + Q_{\alpha\beta} \frac{x^\alpha x^\beta}{|x|^3} + O(|x|^{-3/2})& \end{aligned} \tag{8.3.6}$$

for some constants $Q^{(1)}, Q^{(2)}, Q_\alpha, Q_{\alpha\beta}$, where m and L_α are the constants appearing in (8.3.4).

Proof. Substituting (8.3.4) into (8.3.5) one obtains

$$\begin{aligned}\Delta_{\boldsymbol{\delta}}\Delta_{\boldsymbol{\delta}}\eta &+ \frac{4}{|x|^3}\left[2mx^\alpha - 4m^2\frac{x^\alpha}{|x|} - L^\alpha + 3L_\beta\frac{x^\beta x^\alpha}{|x|^2}\right]\partial_\alpha\Delta_{\boldsymbol{\delta}}\eta \\ &+ \frac{15}{|x|^5}\left[\left(2m - \frac{4m^2}{|x|} + \frac{5}{|x|^2}L_\gamma x^\gamma\right)x^\alpha x^\beta - 2L^{(\alpha}x^{\beta)}\right]\partial_\alpha\partial_\beta\eta \\ &- \frac{1}{|x|^3}\left[10m - \frac{72m^2}{|x|} + \frac{15}{|x|^2}L_\gamma x^\gamma\right]\Delta_{\boldsymbol{\delta}}\eta + \frac{160m^2}{|x|^6}x^\alpha\partial_\alpha\eta + \frac{48m^2}{|x|^6}\eta = O(|x|^{-11/2+\epsilon}). \end{aligned} \quad (8.3.7)$$

Substituting the ansatz, $\eta = \mathfrak{D}|x| + \vartheta$, where $\vartheta = o(|x|^{1/2})$, and collecting lower-order terms in ϑ

$$\Delta_{\boldsymbol{\delta}}\Delta_{\boldsymbol{\delta}}\vartheta = \frac{36\mathfrak{D}m}{|x|^4} + O(|x|^{-9/2}).$$

Using Lemma 21, we obtain

$$\Delta_{\boldsymbol{\delta}}\vartheta = \frac{18\mathfrak{D}m}{|x|^2} - 2Q_\alpha\frac{x^\alpha}{|x|^3} + O(|x|^{-5/2})$$

for some constant-coefficient Q_α. Here we have used that $\Delta_\delta\vartheta = o(|x|^{-3/2})$, thereby excluding constant and $1/|x|$ harmonic terms. Applying Lemma 21 again we find that

$$\vartheta = 18\mathfrak{D}m\ln|x| + Q_\alpha\frac{x^\alpha}{|x|} + Q^{(1)} + \varphi$$

for some constant $Q^{(1)}$, and some function $\varphi = O(|x|^{-1/2})$. Substituting into (8.3.7),

$$\Delta_{\boldsymbol{\delta}}\Delta_{\boldsymbol{\delta}}\varphi = \frac{624\mathfrak{D}m^2}{|x|^5} + (46\mathfrak{D}L_\alpha - 52mQ_\alpha)\frac{x^\alpha}{|x|^6} + O(|x|^{-11/2+\epsilon})$$

implying that

$$\Delta_{\boldsymbol{\delta}}\varphi = \frac{104\mathfrak{D}m^2}{|x|^3} + \frac{1}{2}(23\mathfrak{D}L_\alpha - 26mQ_\alpha)\frac{x^\alpha}{|x|^4} + Q_{\alpha\beta}\frac{x^\alpha x^\beta}{|x|^5} + O(|x|^{-7/2+\epsilon})$$

for some constant-coefficient, tracefree $Q_{\alpha\beta}$. Here we have used the fact that $\Delta_{\boldsymbol{\delta}}\varphi = O(|x|^{-5/2})$ to eliminate constant, $1/|x|$ and $1/|x|^2$ harmonic terms. Integrating up once more, we obtain (8.3.6). □

It is interesting to note the presence of a logarithmically-singular term in (8.3.6) in the non-Killing case, $\mathfrak{D} \neq 0$. On the other hand, if one sets $\mathfrak{D} = 0$ in (8.3.6), then one obtains the asymptotic expansion

$$\eta = Q_\alpha\frac{x^\alpha}{|x|} + Q^{(1)} + \frac{Q^{(2)}}{|x|} + \frac{13}{2}mQ_\alpha\frac{x^\alpha}{|x|^2} + Q_{\alpha\beta}\frac{x^\alpha x^\beta}{|x|^3} + O(|x|^{-3/2}) \quad (8.3.8)$$

for the lapse of a general spacetime Killing vector restricted to a conformally flat initial data set.

8.4 An integral expression for Dain's invariant

In this section we will derive a bulk-integral expression for Dain's invariant —i.e. an integral over $\mathcal{S}$— generalising the expression given in [11] for time symmetric initial data

$$\mathfrak{D} = \frac{1}{16\pi}\int_{\mathcal{S}} X r_{ij} r^{ij} d\mu. \tag{8.4.1}$$

The expression makes clear the fact that the Dain invariant is indeed a coordinate-independent quantity. While the approximate KID set is a global object, being given by the solution of an elliptic PDE defined over the whole of $\mathcal{S}$, the bulk-integral formulation provides a possible means of localising the Dain invariant.

As in [11], we begin by expressing the invariant as follows

$$\begin{aligned}\mathfrak{D} &= -\frac{1}{8\pi}\oint_{\partial\mathcal{S}_\infty} n^k D_k \Delta\left(\mathfrak{D}|x|\right)\, dS\\ &= -\frac{1}{8\pi}\oint_{\partial\mathcal{S}_\infty} n^k D_k \Delta X\, dS\\ &= -\frac{1}{8\pi}\int_{\mathcal{S}} \Delta\Delta X\, d\mu. \end{aligned}\tag{8.4.2}$$

The second line follows from the fact that the contribution to the boundary integral of the lower-order terms in X vanishes, while the third line follows from an application of the divergence theorem. Note that for clarity we are, without loss of generality, restricting to a single asymptotic end. Now, since $\mathcal{P}\circ\mathcal{P}^*(X, X^i) = 0$, we have that

$$2\Delta\Delta X = r^{ij}D_iD_jX - 2r\Delta X - \tfrac{3}{2}D^irD_iX - \tfrac{1}{2}\left(\Delta r + r_{ij}r^{ij}\right)X - D^iD^jH_{ij} + \Delta H_k{}^k + r^{ij}H_{ij} + \bar{H}. \tag{8.4.3}$$

Through integration by parts, one then finds that

$$\begin{aligned}&\int_{\mathcal{S}}\left(r^{ij}D_iD_jX - 2r\Delta X - \tfrac{3}{2}D^iD_iX - \tfrac{1}{2}X\Delta r\right)d\mu\\ &\qquad = -\frac{1}{2}\int_{\mathcal{S}} X\Delta r\, d\mu + \oint_{\partial\mathcal{S}_\infty} n^k\left(r_{kj}D^jX - XD^jr_{kj} - 2rD_kX + \tfrac{1}{2}XD_kr\right)dS. \end{aligned}\tag{8.4.4}$$

Note that boundary integral in the above vanishes, since $r_{ij} = o(|x|^{-5/2})$, $X = o(|x|)$. One also finds that

$$\begin{aligned}\int_{\mathcal{S}} r^{ij}H_{ij}d\mu = &\int_{\mathcal{S}}\left[2r^{ij}(K^k{}_iK_{jk} - KK_{ij})X + r^{ij}(D_jK_{ki} - D_kK_{ij})X^k - \tfrac{1}{2}K^{ij}(D_kr_{ij})X^k\right]d\mu\\ &+ \oint_{\partial\mathcal{S}_\infty} n^k\left(\tfrac{1}{2}r^{ij}K_{ij}X_k + \tfrac{1}{2}rK_{ki}X^i - r_{ik}K^{ij}X_j\right)dS. \end{aligned}\tag{8.4.5}$$

Again, the boundary integral in fact vanishes, since $r_{ij} = o(|x|^{-5/2})$, $K_{ij} = o(|x|^{-3/2})$ and $X = o(|x|^{1/2})$. Finally, denoting by δ the divergence operator on symmetric 2-tensors as in previous chapters, we find that

$$\begin{aligned}
\int_{\mathcal{S}} \bar{H} d\mu &= 2\int_{\mathcal{S}} (Kh^{ij} - K^{ij})\bar{Q}_{ij} + (K^{ki}K^j{}_k - KK^{ij})\bar{\gamma}_{ij} d\mu \\
&= 2\int_{\mathcal{S}} \begin{pmatrix} K^{ki}K^j{}_k - KK^{ij} \\ Kh^{ij} - K^{ij} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -\delta \end{pmatrix} \mathcal{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} d\mu \\
&= 2\int_{\mathcal{S}} \mathcal{P} \begin{pmatrix} K^{ki}K^j{}_k - KK^{ij} \\ D^k(Kh^{ij} - K^{ij}) \end{pmatrix} \cdot \begin{pmatrix} X \\ X_i \end{pmatrix} d\mu - \oint_{\partial\mathcal{S}_\infty} n^k(\mathcal{A}_k + \mathcal{B}_k + \mathcal{C}_k + \mathcal{D}_k) dS, \quad (8.4.6)
\end{aligned}$$

where here the terms $\mathcal{A}_k,\ \mathcal{B}_k,\ \mathcal{C}_k,\ \mathcal{D}_k$ are as in Section 8.1.3, with

$$\begin{aligned}
\gamma^{ij} &\equiv K^{ki}K^j{}_k - KK^{ij} = o(|x|^{-3}), \\
q^{kij} &\equiv D^k(Kh^{ij} - K^{ij}) = o(|x|^{-5/2}).
\end{aligned}$$

Here we have ignored the boundary terms arising from the integration by parts of the divergence operator in the second, since they are easily seen to vanish. Moreover, it can be easily verified that the boundary term in (8.4.6) vanishes. Collecting together (8.4.4)–(8.4.6) and substituting first into (8.4.3) and then into (8.4.2), one obtains the following bulk integral expression for Dain's invariant

$$\begin{aligned}
\mathfrak{D} = \frac{1}{16\pi}\int_{\mathcal{S}} \Big[& X\left(r_{ij}r^{ij} + \tfrac{1}{2}\Delta r + 2r^{ij}\left(K^k{}_i K_{jk} - KK_{ij}\right)\right) \\
&+ X^k\left(r^{ij}\left(D_j K_{ki} - D_k K_{ij}\right) - \tfrac{1}{2}K^{ij}D_k r_{ij}\right) \\
&-2\begin{pmatrix} X \\ X_k \end{pmatrix} \cdot D\Phi \begin{pmatrix} K^k{}_i K_{jk} - KK_{ij} \\ -\Delta(Kh_{ij} - K_{ij}) \end{pmatrix} \Big] d\mu. \quad (8.4.7)
\end{aligned}$$

Here we are using

$$\int_{\mathcal{S}} D^k D^j H_{ij} d\mu = \int_{\mathcal{S}} \Delta H_k{}^k d\mu = 0,$$

which follows from the divergence theorem and the fact that $H_{ij} = o(|x|^{-2})$. Note that in the integral formula (8.4.7), there do not appear any derivatives of the lapse and shift of the approximate Killing vector. Note also that if one sets $K_{ij} = 0$ so that the data is time symmetric, then it follows from the Hamiltonian constraint that $r = 0$, and the above formula thereby reduces to (8.4.1), as given by Dain. An interesting property of (8.4.7) is that, while the Dain invariant was defined by the asymptotics of the lapse alone, the representation of the invariant of the bulk integral requires terms involving the shift X_i. This is perhaps not so surprising since in the approximate Killing equation the equations for the lapse and shift are highly coupled.

8.5 Concluding remarks

We have shown that the existence of approximate Killing vectors extends to a large class of asymptotically Euclidean initial data with non-vanishing extrinsic curvature. Following Dain's discussion in [11], we can then define a geometric invariant $\mathfrak{D}_{(k)}$ on each asymptotically Euclidean end given by the leading coefficient in the appropriate asymptotic expansion. The vanishing of any one of the $\mathfrak{D}_{(k)}$ characterises stationarity of the initial data.

Further work could involve the construction of approximate KID sets on hyperboloidal, rather than asymptotically-Euclidean, hypersurfaces, or the construction of approximate KID sets which generalise the rotational Killing vectors, for instance. It would also be of interest to explore the dynamics of the approximate KID. If a propagation equation for the approximate KID can be found, one may be able to use Dain's invariant to quantify the deviation from stationarity of a generic asymptotically Euclidean initial data set. One could also repeat the analysis of this chapter using the mixed-order system $D\Phi \circ D\Phi^*(X, X_i) = 0$ as the approximate KID equation, rather than $\mathcal{P} \circ \mathcal{P}^*(X, X_i) = 0$, noting that the inconsistency of the units —see Remark 57— could be fixed simply by introducing a characteristic length scale, for instance. In order to use $D\Phi \circ D\Phi^*(X, X_i) = 0$, one would again need a Fredholm theory for Douglis–Nirenberg elliptic systems on weighted Sobolev spaces. Since $D\Phi \circ D\Phi^*(X, X_i) = 0$ are the Euler–Lagrange equations for the action

$$\int_{\mathcal{S}} \|D\Phi^*(X, X_i)\|^2 \; d\mu,$$

and since the Einstein field equations are equivalent to the ADM equations (see the discussion in Section 1.2)

$$\frac{\partial}{\partial t}\begin{pmatrix} \gamma_{ij} \\ K_{ij} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} D\Phi^* \begin{pmatrix} X \\ X_i \end{pmatrix},$$

it may be possible to interpret the approximate KID set as defining a *minimal distortion gauge* —i.e. a timelike direction in which one should evolve the Einstein field equations in order to minimise the change of the 3−metric and extrinsic curvature, $(\gamma_{ij},\ K_{ij})$. This requires further work to be made precise.

Chapter 9

Conclusions and Outlook

The results of this thesis concern two distinct, though tangentially related problems, namely (1): the construction of initial data, and (2): the construction of approximate spacetime Killing symmetries. Our approach to problem (1) is a perturbative one, ultimately relying on an application of the Implicit Function Theorem (IFT), and is based on the work of A. Butscher [28,29] in which solutions of the Einstein constraints were constructed as non-linear perturbations of a background initial data set, via the so-called the Extended Constraint Equations (ECEs). The method differs fundamentally from previously existing methods, such as the "Conformal Method" of Lichnerowicz–Bruhat–York, in its choice of free and determined fields. In particular, certain components of the electric and magnetic parts of the Weyl curvature may be prescribed at the outset. As such, the method offers a potentially new viewpoint on the problem of identifying the degrees of freedom of the gravitational field, which as of yet has not been satisfactorily resolved. Along the way, it was seen that KID sets naturally arise as obstructions in (a variant of) the method, which is natural given their connection to problems of linearisation stability. It is here that there connection is drawn with problem (2), in which approximate Killing vectors are constructed in terms of approximate KID sets, generalising the approach of S. Dain in [11]. The chief motivation in studying the ECEs, however, lies in the fact that they are a simplified version of the Conformal Constraint Equations (CCEs), and the latter offer a potential avenue for the construction of initial data for which (semi-)global existence result of [35] holds. An understanding of the CCEs has potential application to a variety of questions concerning the asymptotic properties of spacetimes, such as issues of stability and the regularity of the conformal boundary, $\mathscr{I}$.

In Chapter 3, we described the origin of the CCEs as the constraint equations implied by the CFEs of H. Friedrich, and their interpretation as a conformally-covariant version of the Einstein constraint equations. Various reductions of the CCEs were given, the most important of which being the ECEs which are obtained simply by imposing a trivial intrinsic and extrinsic conformal rescaling —i.e. setting $\Omega = 1$, $\sigma = 0$. Of particular importance for the remainder of the thesis were the integrability conditions satisfied by the CCEs, and their reduction for the ECEs, which were explored in Sections 3.2.3 and 3.3.2. A subset of the integrability conditions had been previously given in [33]. The integrability conditions were seen to be crucial in the sufficiency part of the Friedrich– Butscher method (and its generalisation to the full CCEs in Chapter 7). Heuristically speaking, the integrability conditions preserve the information which would otherwise seemingly be lost in the process of elliptic reduction —i.e. in moving from the original equations (either the ECEs or the CCEs) to the auxiliary elliptic equations. It was pointed out that one can draw a loose analogy between the Friedrich–Butscher method and the process of hyperbolic reduction of the Einstein field equations.

In Chapter 4, we outlined the Friedrich–Butscher method emphasising the key structural properties of the ECEs which enable such an approach, including the overdetermined-ellipticity of the Codazzi operator when considered to act on $\mathscr{S}^2_0(\mathcal{S};\boldsymbol{h})$ and the underdetermined-ellipticity of the electromagnetic constraint equations, $\Lambda_i = \bar{\Lambda}_i = 0$. These properties were shown to naturally single out a choice of free and prescribed fields, based on the desire to arrive at an elliptic system of auxiliary equations. Having reduced the problem to solving an elliptic system of PDEs, one can appeal to standard Fredholm theory when applying the IFT. A first discussion of the obstructions to the implementation of the method was given in section 4.1.4. In particular, it was shown that (at least in the case of umbilical background geometry) the existence of either a global conformal Killing vector field or a global tracefree Codazzi tensor field would destroy injectivity of the linearised auxiliary map, and therefore stand in the way of a straightforward application of the IFT. The study of the (non-) existence of tracefree Codazzi tensors naturally leads to the concept of conformal rigidity. A first application of the method for the construction of initial data for cosmological spacetimes was then given, where the background geometries (which comprise an infinite family) correspond to spatially closed and conformally-rigid analogues of initial data for the "$k=-1$" FLRW spacetime. Central to the argument is the fact that, when considered as a system for the appropriate zero quantities, the joint auxiliary-integrability system exhibits certain elliptic structures. The regularity of the initial data so-constructed depends on the regularity of the free data; in particular, the initial data can be made arbitrarily smooth by restricting the free data. Finally, it was noted that for such background initial data, the set of smooth free data can be explicitly parametrised by means of the Gasqui–Goldschmidt–Beig complex, which uses the results of [58]. It is reasonable to expect that the smoothness restriction is unnecessary —one needs a generalisation of the Splitting Lemma to elliptic complexes. It remains unclear whether the existence of potential obstructions can be dealt with through a more involved argument, as in [28, 29]. This is one direction in which further study could be directed.

In Chapter 5, the Friedrich–Butscher method was explored more systematically than in Chapter 4. In particular, we considered closed time symmetric, but otherwise general, background initial data sets and established conditions, (C1)–(C4), under which such an initial data set admits non-linear perturbative solutions of the ECEs of the form considered in Chapter 4. In order to deal with the additional coupling of the equations as a result of non-trivial intrinsic curvature, we made use of gauge-fixing procedure which generalises the classical approach of de Turck. More precisely, it was shown that the linearised auxiliary equations are simplified if the gauge-fixing vector is adapted to the electric part of the Weyl curvature in a specific way. Another interesting feature of the analysis is that the integrability condition for the V_{ij} zero quantity featured not only in the sufficiency argument, but also (in its linearised form) in the analysis of the linearised auxiliary equations. In the process, a family of first-order elliptic operators $\mathcal{P}^{(\alpha)}$ was identified, of which two, $\mathcal{P}^{(0)}$ and $\mathcal{P}^{(1)}$ feature prominently in the ECEs and their integrability conditions. At the end of the chapter, a preliminary investigation of the conditions (C1)–(C4) was given for the cases of positive and negative cosmological constants. The results are only partial, but are suggestive of further development. In particular, it was seen in the case of negative cosmological constant that if the curvature of the background metric is suitably pinched, then conditions (C1)–(C3) are satisfied, leaving only (C4) — see Corollary 3. Though condition (C4) requires more analysis, it is straightforward to see that such metrics at least admit no non-trivial conformal Killing vector fields (see section 4.1.4); this is a prerequisite for condition (C4). The sub-family of time symmetric conformally-rigid hyperbolic metrics, considered in Chapter 4, are examples of negatively-pinched metrics which moreover satisfy condition (C4). As such, the main result of this Chapter, Theorem 3, constitutes a partial generalisation of Theorem 2 of Chapter 4. Allowing for non-trivial mean extrinsic curvature, K, (i.e. generalising from time symmetric data to umbilical data) requires some additional "book-keeping" but is anticipated to be straightforward. It would interesting to see whether any of the sufficient conditions (which would presumably now involve K) trivialise for particular choices of K. It would also be interesting to see whether the conditions (C1)–(C4) are necessary for $D_v\widetilde{\Psi}$ to be invertible, in addition to being sufficient.

The purpose of Chapter 6 was to propose an alternative to the Friedrich–Butscher method in which the ECEs are studied as a mixed-order system of PDEs. This approach makes use of the fact that the Codazzi–Mainardi equation can be considered a first-order elliptic equation —with principal part $\mathcal{P}^{(0)}$ —for the appropriate choice of fields, as suggested by the work of Chapter 5. In addition to this, a new gauge-fixing procedure was used for the Gauss–Codazzi equation; the new procedure and that of de Turck can both be considered special cases of a more general procedure which is outlined A.5 of the Appendix and which appears to be new. The motivation for the new gauge-fixing is threefold: firstly, it effects a useful decoupling of the linearised auxiliary equations; secondly, it greatly simplifies the sufficiency argument; thirdly, the procedure leaves the linearisation of the *trace* of the Gauss–Codazzi equation unchanged, and as a result the KID equations are a subsystem of the adjoint-linearised auxiliary equations. Hence, among the potential obstructions to application of the IFT are the KID sets. In particular, in the case of an umbilical background initial data set, static potentials enter into the cokernel. The proposed method was then applied to closed time symmetric background initial data, as in Chapter 5, and (A1)–(A3) were identified as being sufficient conditions for its implementation. In doing so, we require no assumptions on the kernel of either the Yano Laplacian or the operator $\mathcal{P}^{(1)}$, and so the result (Theorem 6) is an improvement on Theorem 3 of Chapter 5. Further work would involve trying to weaken the conditions further, and obtaining a complete geometric characterisation of the cokernel. Note that for closed $\mathcal{S}$, the Atiyah–Singer index theorem implies that in order for $D_v\widetilde{\Psi}$ to be invertible, it is sufficient to establish either injectivity or surjectivity. In order to apply the method to asymptotically-Euclidean manifolds, one needs a Fredholm theory for Douglis–Nirenberg systems on weighted Sobolev spaces. It is not clear whether such a theory already exists in the literature. It was remarked that the streamlined method proposed in the previous chapter seems to be unsuitable for application to the full CCEs on manifolds with boundary, since the Codzzi–Mainardi equation degenerates at the boundary (on which $\Omega = 0$) —see Remark 39.

In Chapter 7, we returned to the full CCEs. By extending the method of Chapter 4, it was shown that CCEs admit a second-order elliptic reduction for a given choice of free and determined fields. In particular, the trace part of L_{ij}, denoted θ, and the scalar part in the Helmholtz decomposition of L_i, denoted φ, were prescribed as free data. It was shown that (θ, φ) admit a natural interpretation as conformal gauge-source functions, fixing the intrinsic and extrinsic conformal freedom that is built into the CCEs. The auxiliary-integrability system can then be used to derive a system of elliptic equations for the zero quantities, allowing a sufficiency argument in the spirit of the Friedrich–Butscher method to, in principle, be carried out. As a first application, we considered the closed conformally-rigid hyperbolic initial data of Chapter 4, considered now as solutions to the full CCEs by setting $\mathring{\Omega} = 1,\ \mathring{\sigma} = 0$. It was shown that such background solutions admit non-linear perturbative solutions of the CCEs, which can be related to the solutions of Chapter 4 through a conformal transformation. The sufficiency argument proved to be the most difficult part of the analysis, on account of the sheer size and complexity of the CCEs compared to the ECEs. The results of this chapter should be considered a proof of concept; the task of extending the analysis to manifolds with boundary (which is, of course, the ultimate aim) requires a lot of work, though a heuristic discussion in this direction was given in Section 7.3. In future work, the natural starting point would be the "umbilical CCEs" given in Section 3.3.1.

Finally, in Chapter 8, we considered problem (2). It was shown that the notion of an approximate KID set, defined by Dain on a time symmetric initial data set, generalises to generic asymptotic-Euclidean initial data. The approximate KID set has the property that it reduces to an exact stationary Killing vector field, whenever such a vector fields exists. Having constructed the approximate Killing vector, one can read off an invariant $\mathfrak{D}_{(k)}$, the *Dain invariant*, from the asymptotics of the lapse function at each asymptotic end. The vanishing of the Dain invariant (one any one of the ends) characterises stationarity of the initial data set. A more detailed asymptotic expansion to $\mathcal{O}(|x|^{-3/2})$ was given for the lapse of an approximate KID on a conformally flat initial data set, in terms of the ADM mass m and linear momentum L_α. Since the approximate KID equation is linear, one has the freedom to rescale the approximate KID set by any non-zero constant. Therefore, in order to attribute meaning to the magnitude of $\mathfrak{D}_{(k)}$, one has to find a meaningful way of fixing the scale. One option is to try to construct evolution equations for the approximate KID set —i.e. equations which, if used to evolve the approximate KID set of a given initial hypersurface, $\mathcal{S}_0$, preserve the approximate KID equation. Then, one would only need to choose a scale for the approximate KID set on $\mathcal{S}_0$, and the magnitude of $\mathfrak{D}_{(k)}$ on the subsequent spacetime development would be determined and would, in a sense, quantify departure from stationarity of the underlying initial data set. Such an invariant could have applications to stability problems, in which it would be beneficial to have a canonical way of quantifying a gravitating system's progress towards settling down to equilibrium. In order to have applications to spacetimes such as Kerr, it would be interesting to try to extend the given results to spacetimes with an inner boundary, such as a trapped surface. To do so would we must study the approximate KID equation as an elliptic BVP; presumably, one would choose boundary conditions which guarantee that the (exact) KID equations are satisfied on the inner boundary. These conditions would clearly be overdetermined, and therefore one expects some kind of geometric constraint on the boundary, such as a trapping condition, in order for there to exist a non-trivial solution. Of course, the boundary conditions must also be set up in such a way that the resulting BVP is well-posed —e.g. the Lopatinski–Shapiro conditions would need to be satisfied. It would also be interesting to construct approximate Killing vectors which generalise the rotational Killing vectors, rather than the stationary Killing vectors. Such an approximate KID set would require a shift part that blows up like $|x|$, and so one would have to use a different ansatz than that considered here. This has potential applications to the black hole uniqueness problem; I am grateful to Prof. Chruściel for this suggestion.

Appendix A

A.1 Conformal rigidity and hyperbolic manifolds

Here we give the proofs of Proposition 3 from Section 4.2.2, and Lemma 8 from Section 4.4.1.

Proposition. Let $(\mathcal{S}, \mathring{\boldsymbol{h}})$ be a closed hyperbolic manifold, then

$$\ker\{\mathring{\mathcal{D}} : \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}}) \to \mathcal{J}(\mathcal{S})\} = \ker\, H \cap \ker\, \mathring{\delta}.$$

Hence, if $\mathring{\boldsymbol{h}}$ is conformally-rigid then it admits no non-trivial tracefree Codazzi tensors.

Proof. On one hand, recall that

$$\ker\, \mathring{\mathcal{D}} = \ker\, \mathring{\mathcal{R}} \cap \ker\, \mathring{\delta} = \ker\{\mathring{\mathcal{R}} : \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}}) \to \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})\}.$$

On the other hand,

$$\ker\, H \cap \ker\, \mathring{\delta} = \ker\{H : \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}}) \to \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})\}.$$

Accordingly, we aim to show that

$$\ker\{\mathring{\mathcal{R}} : \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}}) \to \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})\} = \ker\{H : \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}}) \to \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})\}.$$

First note that for general $\mathring{\boldsymbol{h}}$ and $\boldsymbol{\eta} \in \mathscr{S}^2(\mathcal{S})$,

$$\mathring{D}_i \mathring{\Delta} \eta_{jk} = \mathring{\Delta} \mathring{D}_i \eta_{jk} - \mathring{r}_i{}^l \mathring{D}_l \eta_{jk} + \eta_k{}^l \mathring{D}_m r_i{}^m{}_{jl} + \eta_j{}^l \mathring{D}_m r_i{}^m{}_{kl} + 2 r_{imkl} \mathring{D}^m \eta_j{}^l + 2 r_{imjl} \mathring{D}^m \eta_k{}^l.$$

For a $3-$dimensional Einstein manifold, we have $\mathring{r}_{ijkl} = \frac{1}{3} \mathring{r} \mathring{h}_{k[i} \mathring{h}_{j]l}$, and so we obtain

$$\mathring{D}_i \mathring{\Delta} \eta_{jk} = \mathring{\Delta} \mathring{D}_i \eta_{jk} - \mathring{r} \mathring{D}_{(i} \eta_{jk)} + \tfrac{2}{3} \mathring{r} \mathring{h}_{i(j} \mathring{\delta}(\boldsymbol{\eta})_{k)}.$$

Hence, for $\eta_{ij} \in \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})$,

$$\mathring{\mathcal{R}} \circ \mathring{\Delta}(\boldsymbol{\eta})_{ijk} = \mathring{\Delta} \circ \mathring{\mathcal{R}}(\boldsymbol{\eta})_{ijk}$$

Since we also have vanishing of the Cotton-York tensor, $\mathring{\mathcal{H}}_{ij} = 0$, we find that for $\boldsymbol{\eta} \in \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})$,

$$H(\boldsymbol{\eta})_{ij} = -\tfrac{1}{2} \mathring{\mathcal{R}} \circ \mathring{\Delta}(\boldsymbol{\eta})_{ij} + \tfrac{1}{6} \mathring{r} \mathring{\mathcal{R}}(\boldsymbol{\eta})_{ij} = \tfrac{1}{2} \left(-\mathring{\Delta} + \tfrac{1}{3} \mathring{r} \right) \mathring{\mathcal{R}}(\boldsymbol{\eta})_{ij}$$

If $\boldsymbol{\eta} \in \ker\{\mathring{\mathcal{R}} : \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}}) \to \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})\}$ then clearly $\boldsymbol{\eta} \in \ker\{H : \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}}) \to \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})\}$. For the converse, first note that by a lengthy calculation, we have that for $\boldsymbol{\eta} \in \mathscr{S}^2_0(\mathcal{S}; \mathring{\boldsymbol{h}})$,

$$\begin{aligned}\mathring{\mathcal{R}}^2(\boldsymbol{\eta})_{ij} &= -\mathring{\Delta}\eta_{ij} + \tfrac{3}{2}\mathring{D}_{\{i}\mathring{\delta}(\boldsymbol{\eta})_{j\}} + \mathring{r}_{(i}{}^{k}\eta_{j)k} - \mathring{r}_{ikjl}\eta^{kl} \\ &= \left(-\mathring{\Delta} + \tfrac{1}{2}\mathring{r}\right)\eta_{ij} + \tfrac{3}{2}\mathring{D}_{\{i}\mathring{\delta}(\boldsymbol{\eta})_{j\}},\end{aligned}$$

where the second equality holds for $\mathring{\boldsymbol{h}}$ Einstein. Then, restricting to $\boldsymbol{\eta} \in \mathscr{S}_{TT}(\mathcal{S}; \mathring{\boldsymbol{h}})$, equation $H(\boldsymbol{\eta})_{ij} = 0$ is equivalent to

$$\left(\mathring{\mathcal{R}}^2 - \tfrac{2}{3}\mathring{r}\right)\mathring{\mathcal{R}}(\boldsymbol{\eta})_{ij} = 0.$$

Now, contracting with $\mathring{\mathcal{R}}(\boldsymbol{\eta})_{ij}$ and integrating by parts using $\mathring{\mathcal{R}}^* = \mathring{\mathcal{R}}$, one finds that

$$\|\mathring{\mathcal{R}}^2(\boldsymbol{\eta})\|^2_{L^2} - \tfrac{2}{3}\mathring{r}\|\mathring{\mathcal{R}}(\boldsymbol{\eta})\|^2_{L^2} = 0.$$

Since $\mathring{r} < 0$, we see then that $\mathring{\mathcal{R}}^2(\boldsymbol{\eta})_{ij} = 0$. Contracting with η^{ij} and integrating by parts once more we find that $\mathring{\mathcal{R}}(\boldsymbol{\eta})_{ij} = 0$, as required. □

Lemma. Let $(\mathcal{S}, \mathring{\boldsymbol{h}})$ be a conformally flat manifold, then $\operatorname{Im}\mathring{L} \subseteq H$.

Proof. Recall the conformal transformation law for the Cotton–York tensor:

$$\mathcal{H}[\theta^2\boldsymbol{h}]_{ij} = \theta^{-1}\mathcal{H}[\boldsymbol{h}]_{ij} \qquad \text{for} \qquad \theta > 0.$$

Now, by diffeomorphism invariance, we have $\varphi^*\mathcal{H}[\boldsymbol{h}]_{ij} = \mathcal{H}[\varphi^*\boldsymbol{h}]_{ij}$ for any diffeomorphism φ. Let φ_τ be the one-parameter-family of diffeomorphisms generated by a given vector field, $\boldsymbol{X}$. Combining the above, we see that

$$\begin{aligned}\mathcal{L}_{\boldsymbol{X}}\mathring{\mathcal{H}}_{ij} = \frac{d}{d\tau}\varphi^*_\tau\mathcal{H}[\mathring{\boldsymbol{h}}]_{ij}\bigg|_{\tau=0} = \frac{d}{d\tau}\mathcal{H}[\varphi^*_\tau\mathring{\boldsymbol{h}}]_{ij}\bigg|_{\tau=0} &= H[\mathcal{L}_{\boldsymbol{X}}\mathring{\boldsymbol{h}}]_{ij} \\ &= H[\mathring{L}(\boldsymbol{X})]_{ij} + \frac{2}{3}H[\mathring{\delta}(\boldsymbol{X})\mathring{\boldsymbol{h}}]_{ij} \\ &= H[\mathring{L}(\boldsymbol{X})]_{ij} + \frac{2}{3}\frac{d}{d\tau}\mathcal{H}[(1+\tau\mathring{\delta}(\boldsymbol{X}))\mathring{\boldsymbol{h}}]_{ij}\bigg|_{\tau=0} \\ &= H[\mathring{L}(\boldsymbol{X})]_{ij} + \frac{2}{3}\frac{d}{d\tau}(1+\tau\mathring{\delta}(\boldsymbol{X}))^{-1/2}\mathring{\mathcal{H}}_{ij}\bigg|_{\tau=0} \\ &= H[\mathring{L}(\boldsymbol{X})]_{ij} - \frac{1}{3}\mathring{\delta}(\boldsymbol{X})\mathring{\mathcal{H}}_{ij}\end{aligned}$$

Hence,

$$H \circ \mathring{L}(\boldsymbol{X})_{ij} = \mathcal{L}_{\boldsymbol{X}}\mathring{\mathcal{H}}_{ij} + \tfrac{1}{3}\mathring{\delta}(\boldsymbol{X})\mathring{\mathcal{H}}_{ij},$$

which can, of course, be checked by explicit computation. If $\mathring{\boldsymbol{h}}$ is conformally flat then we find that $H[\mathring{L}(\boldsymbol{X})]_{ij} = 0$. Since this holds for all $\boldsymbol{X}$, we see that $\operatorname{Im}\mathring{L} \subseteq \ker H$. □

A.2 Fredholm Theory for Douglis–Nirenberg elliptic operators

Here we give a sketch proof of Theorem 5, stated in Section 6.2.1. Recall that the Banach spaces $\mathcal{B}^l_1$, $\mathcal{B}^l_2$ are defined by

$$\mathcal{B}^l_1 \equiv H^{l+t_1} \times \cdots \times H^{l+t_N}, \qquad \mathcal{B}^l_2 \equiv H^{l-s_1} \times \cdots \times H^{l-s_2}.$$

$$P(x, D)_{\mu\nu} u^{\nu} = F(x)_{\mu} \tag{A.2.1}$$

Theorem. Let $(\mathcal{S}, \boldsymbol{h})$ be a smooth closed Riemannian manifold and $P(x, D)_{\mu\nu}$ a Douglis–Nirenberg operator with weights s_{μ} and t_{ν}. Then, given a non-negative integer l, there exists a positive constant C_l such that, if $u^{\sigma} \in H^{t_{\sigma}}$ is a solution to (A.2.1) with $F_{\mu} \in C^{l-s_{\mu}}$, then $u^{\sigma} \in H^{l+t_{\sigma}}$ and

$$\|u^{\sigma}\|_{H^{l+t_{\sigma}}} \leq C_l \left(\sum_{\mu=1}^{N} \|F_{\mu}\|_{H^{l-s_{\mu}}} + \sum_{\nu=1}^{N} \|u^{\nu}\|_{H^{1+t_{\nu}-1}} \right). \tag{A.2.2}$$

It follows that the operator $P : B^l_1 \to B^l_2$ is Fredholm. More specifically,

(i) ker $P|_{\mathcal{B}^l_1}$ is closed and finite dimensional, and Im $P|_{\mathcal{B}^l_1}$ is closed in $\mathcal{B}^l_2$;

(ii) coker P is closed in B^l_2 and finite-dimensional, and is isomorphic to ker $P^*|_{\mathcal{B}^l_2}$.

Proof. (*Sketch*) We follow the proof of Theorem 2.1, Appendix II of [14]. We cover $\mathcal{S}$ with a finite number of coordinate charts indexed by I and take a smooth partition of unity, $0 \leq \phi_I \leq 1$, subordinate to the cover. Define for each ν and each I the function $u^{\nu}_I \equiv \phi_I u^{\nu}$. Clearly $u^{\nu}_I \in \tilde{H}^{t_{\nu}}$ for each I and each ν, and by Theorem 4, we automatically have $u^{\nu}_I \in \tilde{H}^{l+t_{\nu}}$, and hence $u^{\nu} \in H^{l+t_{\nu}}$ for each ν.

Now, denote the principal part of $P(x, D)_{\mu\nu}$ by

$$p(x, D)_{\mu\nu} = \sum_{|\rho|=s_{\mu}+t_{\nu}} A_{\mu\nu,\rho} D^{\rho},$$

with ρ a multi-index. Let us first consider the operator $\tilde{p}(x, \partial)_{\mu\nu} \equiv p(x, \partial)_{\mu\nu}$. Application of the estimate (6.2.4) yields

$$\|u^{\sigma}_I\|_{l+t_{\sigma}} \leq K_I \left(\sum_{\mu=1}^{N} \|\tilde{p}(x, \partial)_{\mu\nu} u^{\nu}_I\|_{\tilde{H}^{l-s_{\mu}}} + \sum_{\mu=1}^{N} \|u^{\mu}_I\|_{\tilde{L}^2} \right)$$

for each I and each σ. Note that (modifying constants where necessary) we can immediately replace $\|\cdot\|_{\tilde{H}^{l-s_{\mu}}}$ norms with $\|\cdot\|_{H^{l-s_{\mu}}}$ norms. Hence,

$$\|u^{\sigma}\|_{l+t_{\sigma}} \leq \sum_I \|u^{\sigma}_I\|_{l+t_{\sigma}} \leq \kappa \sum_I \left(\sum_{\mu=1}^{N} \|\tilde{p}(x, \partial)_{\mu\nu} u^{\nu}_I\|_{l-s_{\mu}} + \sum_{\mu=1}^{N} \|u^{\mu}_I\|_{L^2} \right) \tag{A.2.3}$$

with $\kappa = \max_I K_I$. Now, it easily verified that, schematically,

$$\tilde{p}(x, \partial)_{\mu\nu} u^{\nu}_I - p(x, D)_{\mu\nu} u^{\nu}_I = \sum_{k=0}^{s_{\mu}+t_{\nu}-1} \sum_{|\rho|=k} (\boldsymbol{A}(x) \cdot \boldsymbol{S}(x)^{(k)}_I)_{\mu\nu,\rho} \cdot D^{\rho} u^{\nu}_I$$

where $\boldsymbol{A}$ denotes the components $a_{\mu\nu}$ and $\boldsymbol{S}^{(k)}_I(x)$ are smooth functions obtained algebraically from the k-th derivatives of the transition tensor relating the metrics $\boldsymbol{\delta}$ and $\boldsymbol{h}$ in the chart I. Hence it follows that

$$\|\tilde{p}(x, \partial)_{\mu\nu} u^{\nu}_I\|_{H^{l-s_{\mu}}} \leq \|p(x, D)_{\mu\nu} u^{\nu}_I\|_{H^{l-s_{\mu}}} + C_l \|A_{\mu\nu}\|_{C^{l-s_{\mu}}} \|u^{\nu}_I\|_{H^{l+t_{\nu}-1}} \tag{A.2.4}$$

for some constant $C_1 > 0$. On the other hand, recalling that $u^{\nu}_I \equiv \phi_I u^{\nu}$ and using the product rule it follows that

$$\begin{aligned}\|D^\rho u_I^\nu\|_{H^{l-s_\mu}} &\leq \sum_{k=0}^{|\rho|=s_\mu+t_\nu} \sum_{\varrho\leq\rho,|\rho|-|\varrho|=k} \max_I \|\phi_I\|_{C^{l-k+t_\nu}} \|D^\varrho u^\nu\|_{H^{l-s_\mu}} \\ &\leq \max_I \|\phi_I\|_{C^{l-s_\mu}} \|D^\rho u^\nu\|_{H^{l-s_\mu}} + \sum_{k=1}^{s_\mu+t_\nu} \sum_{|\rho|-|\varrho|=k} \max_I \|\phi_I\|_{C^{l-k+t_\nu}} \|D^\rho u^\nu\|_{l-s_\mu},\end{aligned}$$

where by $\varrho \leq \rho$, we mean all tuples $\varrho = (\varrho_1\varrho_2\cdots\varrho_n)$ for which $\varrho_j \leq \rho_j$ for all $j = 1, \cdots n$ —here, n the dimension of $\mathcal{S}$. It follows then that

$$\|p(x,D)_{\mu\nu} u_I^\nu\|_{H^{l-s_\mu}} \leq C_2 \left(\|p(x,D)_{\mu\nu} u^\nu\|_{H^{l-s_\mu}} + \|A_{\mu\nu}\|_{C^{l-s_\mu}} \|u^\nu\|_{H^{l+t_\nu-1}}\right) \tag{A.2.5}$$

for some constant $C_2 > 0$. Collecting together (A.2.3), (A.2.4) and (A.2.5) we find that

$$\|u^\sigma\|_{H^{l-s_\sigma}} \leq C_3 \left(\sum_{\mu=1}^N \|p(x,D)_{\mu\nu} u^\nu\|_{H^{l-s_\mu}} + \sum_{\nu=1}^N \|u^\nu\|_{H^{l+t_\nu-1}} \right) \tag{A.2.6}$$

for some $C_3 > 0$. Finally, recalling that $A_{\mu\nu} \in C^{l-s_\mu}$, an argument analogous to that for (A.2.4) yields that

$$\|p(x,D)_{\mu\nu} u^\nu\|_{H^{l-s_\mu}} \leq \|P(x,D)_\mu u^\nu\|_{H^{l-s_\mu}} + C_4 \|a_{\mu\nu}\|_{C^{l-s_\mu}} \|u^\nu\|_{H^{l+t_\nu-1}},$$

for some $C_4 > 0$. When combined with (A.2.6) this gives the desired estimate.

Proof of (i): Consider the unit ball in ker $P|_{\mathcal{B}_1^l}$. Given any sequence therein, there exists a Cauchy subsequence in $\mathcal{B}_1^{l-1}$ by Rellich–Kondrakov, which by the above estimate is also Cauchy in $\mathcal{B}_1^l$. Hence it follows that the unit ball in ker $P|_{\mathcal{B}_1^l}$ is compact, implying that ker $P|_{\mathcal{B}_1^l}$ is finite-dimensional. This also implies that $\mathcal{B}_1^l =$ ker $P \oplus \mathcal{C}^l$ with $\mathcal{C}^l = (\text{ker } P)^\perp$ a sub-Banach space of $\mathcal{B}_1^l$. When restricted to act on $\mathcal{C}^l$, P is injective and, by a standard argument[1] the estimate (A.2.2) can be improved to

$$\|u^\sigma\|_{H^{l+t_\sigma}} = C_l \sum_{\mu=1}^N \|P_{\mu\nu}(x,D)u^\nu\|_{H^{l-s_\mu}}. \tag{A.2.7}$$

Note also that Im $P|_{\mathcal{C}^l} =$ Im $P|_{\mathcal{B}_1^l}$. Hence, given a Cauchy sequence $w_{(k)}$ in Im $P \subset \mathcal{B}_2^l$, the (unique) pre-images in $\mathcal{C}^l$ form a Cauchy sequence, obtaining some limit, $u = (u^1, \cdots, u^N)$, say. Then, since $P : \mathcal{C}^l \to \mathcal{B}_2^l$ is continuous, clearly $w_{(k)}$ converges to $P(u)$, and hence Im P is closed.

Proof of (ii): The proof is classical —again, see the proof of Theorem 2.1, Appendix II of [14]. □

A.3 Sectional curvatures and curvature pinching

Here we recall the definition of sectional curvature, and its relationship to the eigenvalues of the Riemann and Ricci curvatures for 3-dimensional Riemannian manifolds.

Definition 16. The *sectional curvature* at of the plane spanned by $X^i, Y^i \in T_p\mathcal{S}$, at $p \in \mathcal{S}$, is given by

$$\sigma_p(\boldsymbol{X},\boldsymbol{Y}) \equiv \frac{\mathrm{R}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{X},\boldsymbol{Y})}{\|\boldsymbol{X}\|^2\|\boldsymbol{Y}\|^2 - |\langle \boldsymbol{X},\boldsymbol{Y}\rangle|^2}.$$

[1]See the proof of Theorem 2.1, Appendix II of [14], for example.

Recall that we can think of the $(2,2)$ Riemann curvature as an endomorphism $\Lambda^2(T\mathcal{S})$. In dimension three, the Riemann curvature has three eigentensors (2−forms), T_I^{ij} for $I = 1,2,3$ say. Denote the corresponding eigenvalues by $\lambda_I,\ I = 1,2,3$. Moreover, the eigentensors may written in the form of *bi-vectors* —i.e. $T_I^{ij} = V_I^{[i}W_I^{j]}$ for some $V_I^i, W_I^i \in T_p\mathcal{S},\ I = 1,2,3$. Without loss of generality we can take $\boldsymbol{V}_I, \boldsymbol{W}_I$ to be orthonormal, in which case it is clear that

$$\sigma_p(\boldsymbol{V}_I, \boldsymbol{W}_I) = \lambda_I,$$

for each $I = 1,2,3$. It follows then that

$$\min_{I=1,2,3} \lambda_I \leq \sigma_p(\boldsymbol{X}, \boldsymbol{Y}) \leq \max_{I=1,2,3} \lambda_I$$

for all X^i, Y^i. Now, let X^i, Y^i, Z^i denote the eigenvectors or the $(1,1)$ Ricci curvature, with eigenvalues μ_1, μ_2, μ_3, say —i.e.

$$r_i{}^j X^i = \mu_1 X^j, \qquad r_i{}^j Y^i = \mu_2 Y^j, \qquad r_i{}^j Z^i = \mu_3 Z^j.$$

Then, using the Kulkarni–Nomizu decomposition, we find that $X^{[i}Y^{j]},\ Y^{[i}Z^{j]},\ Z^{[i}X^{j]}$ are precisely the eigentensors of the Riemann curvature endomorphism. Hence, up to reordering, the eigenvalues λ_I are given by

$$\lambda_1 = \mu_2 + \mu_3 - \mu_1, \qquad \lambda_2 = \mu_3 + \mu_1 - \mu_2, \qquad \lambda_3 = \mu_1 + \mu_2 - \mu_3.$$

A.4 The kernels of $\mathcal{P}^{(\alpha)}$ on closed Einstein manifolds

Here we give the proof of Proposition 15 from Section 5.3.3.

Proposition. Suppose $(\mathcal{S}, \boldsymbol{h})$ is a closed Einstein manifold with (constant) scalar curvature $r = 2\lambda$, normalised to $\lambda = -3, 0, 3$. Then,

(i) For $\lambda = -3$ and $\alpha \geq 0$, $\ker \mathcal{P}^{(\alpha)} = \mathbf{C} \oplus \{\mathbf{0}\}$,

(ii) For $\lambda = 0$,

- (a) $\ker \mathcal{P}^{(0)} = \mathbf{C} \oplus \mathbf{c}$,
- (b) $\ker \mathcal{P}^{(\alpha)} = \mathbf{C} \oplus \mathbf{p}$ for $\alpha > 0$,

(iii) For $\lambda = 3$,

- (a) $\ker \mathcal{P}^{(0)} = \{\mathbf{0}\} \oplus \mathbf{c}$,
- (b) $\ker \mathcal{P}^{(\alpha)} = \{\mathbf{0}\} \oplus \{\mathbf{0}\}$ for $\alpha > 0$, unless $(\mathcal{S}, \boldsymbol{h})$ is isometric to $\mathbb{S}^3$ whereupon

$$\ker \mathcal{P}^{(\alpha)} = \{\mathbf{0}\} \oplus \mathrm{sp}\langle dx_I, I = 1,2,3,4\rangle,$$

where $x_I,\ I = 1,2,3,4$, are the restriction to $\mathbb{S}^3 \subset \mathbb{R}^4$ of the standard Cartesian coordinate functions on $\mathbb{R}^4$.

Proof. Suppose $(\boldsymbol{Y},\boldsymbol{X})$ satisfies $\mathcal{P}^{(\alpha)}(\boldsymbol{Y},\boldsymbol{X})=0$, then we have

$$\begin{aligned}
0 &= \delta(L(\boldsymbol{X})+2\mathcal{R}(\boldsymbol{Y}))_i\\
&= \delta\circ L(\boldsymbol{X})_i + 2\delta\circ\mathcal{R}(\boldsymbol{Y})_i\\
&= \delta\circ L(\boldsymbol{X})_i + \mathrm{curl}\circ\delta(\boldsymbol{Y})_i - 2\epsilon_{iml}r_j{}^lY^{jm}\\
&= \delta\circ L(\boldsymbol{X})_i - \alpha\mathrm{curl}^2(\boldsymbol{X})_i - 2\epsilon_{iml}r_j{}^lY^{jm},
\end{aligned}$$

where the third line uses the identity

$$\delta\circ\mathcal{R}(\boldsymbol{Y})_i = \tfrac{1}{2}\mathrm{curl}\circ\delta(\boldsymbol{Y})_i - \epsilon_{iml}r_j{}^lY^{jm},$$

and the fourth uses the second component of $\mathcal{P}^{(\alpha)}(\boldsymbol{Y},\boldsymbol{X})=0$. Then, contracting with X^i and integrating by parts:

$$0=\int_{\mathcal{S}}\left(\tfrac{1}{2}\|L(\boldsymbol{X})\|^2+\alpha\|\mathrm{curl}(\boldsymbol{X})\|^2+2\epsilon_{iml}r_j{}^lX^iY^{jm}\right)\,d\mu_{\boldsymbol{h}}, \tag{A.4.1}$$

where we are using the fact that $\delta^*=-L$ and $\mathrm{curl}^*=\mathrm{curl}$. When h_{ij} is Einstein, the algebraic terms vanish and we deduce for $\alpha\geq 0$ that $L(\boldsymbol{X})_{ij}=0$.

Case (i): It follows from (A.4.1) that $L(\boldsymbol{X})_{ij}=0$ for $\alpha\geq 0$. Recall that $\mathbf{c}\equiv\ker L=\{\mathbf{0}\}$ in the case $\lambda=-3$ and hence $X_i=0$. Substituting back into $\mathcal{P}^{(\alpha)}(\boldsymbol{Y},\boldsymbol{X})=0$, one obtains $\mathcal{R}(\boldsymbol{Y})_{ij}=0$ and $\delta(\boldsymbol{Y})_i=0$, or equivalently, $\mathcal{D}(\boldsymbol{Y})_{ijk}=0$. That is to say, Y_{ij} is a tracefree Codazzi tensor.

Case (ii): When $\alpha=0$, substituting $L(\boldsymbol{X})_{ij}=0$ one again obtains $\mathcal{D}(\boldsymbol{Y})_{ijk}=0$ and the result follows. On the other hand, in the case $\alpha>0$, we obtain from (A.4.1) not only that $L(\boldsymbol{X})_{ij}=0$ but also that $\mathrm{curl}(\boldsymbol{X})_i=0$, and substitution into $\mathcal{P}^{(\alpha)}(\boldsymbol{Y},\boldsymbol{X})=0$ again yields $\mathcal{D}(\boldsymbol{Y})_{ijk}=0$. Now, since $L(\boldsymbol{X})_{ij}=0$,

$$\begin{aligned}
0 &= \tfrac{1}{2}\delta\circ\delta L(\boldsymbol{X})\\
&= \tfrac{1}{2}\delta\left(\Delta\boldsymbol{X}+\tfrac{1}{3}d(\delta(\boldsymbol{X}))+\mathrm{Ric}(\cdot,\boldsymbol{X}^{\#})\right)\\
&= \tfrac{2}{3}\Delta\delta(\boldsymbol{X})+\tfrac{1}{2}(dr)_iX^i+r^{ij}D_iX_j\\
&= \left(\Delta+\tfrac{1}{2}r\right)\delta(\boldsymbol{X})\\
&= (\Delta+\lambda)\,\delta(\boldsymbol{X}).
\end{aligned} \tag{A.4.2}$$

For $\lambda=0$ we see that $\delta(\boldsymbol{X})$ is constant and, since it integrates to zero, must therefore vanish. It follows then that $D_{(i}X_{j)}=\mathrm{curl}(\boldsymbol{X})_i=0$ and hence that X_i is parallel i.e. $X_i\in\mathbf{p}$.

Case (iii): When $\alpha=0$, as in Case (i), we see that the kernel is equal to $\mathbf{C}\oplus\mathbf{c}$. In this case $\mathbf{C}=\{\mathbf{0}\}$ —see Proposition 2 in Chapter 4.

Now consider $\alpha>0$. Again, (A.4.1) implies $L(\boldsymbol{X})_{ij}=0$ and $\mathrm{curl}(\boldsymbol{X})_i=0$. For $(\mathcal{S},\boldsymbol{h})$ Einstein and not isometric to $\mathbb{S}^3$, all conformal diffeomorphisms are isometries (see Corollary 1 of [95]) and so it follows from $L(\boldsymbol{X})_{ij}=0$ that $\delta(\boldsymbol{X})=0$. Collecting together the above observations, as in Case (i), we see that X_i is parallel (alternatively, note that X_i is a harmonic $1-$form) and therefore vanishes by positivity of the curvature. On the other hand, consider the case where $(\mathcal{S},\boldsymbol{h})$ is isometric to $\mathbb{S}^3$. The operator $(\Delta+\lambda)=(\Delta+3)$ has in this case a $4-$dimensional kernel spanned by the functions x_I, $I=1,2,3,4$ —see [78], for instance. Hence, from (A.4.2) we see that $\delta(\boldsymbol{X})=F$ for some $F\in\mathrm{sp}\langle x_I\rangle$. Now, since the first de Rham cohomology of $\mathbb{S}^3$ vanishes, $\mathrm{curl}(\boldsymbol{X})_i=0$ implies $X_i=(df)_i$ for some

$f : \mathcal{S} \to \mathbb{R}$. In terms of f, we have that

$$\Delta f = \delta(\boldsymbol{X}) = F.$$

Since F satisfies $(\Delta+3)F = 0$, it is clear that the solution f is given (up to a constant[2]) by $f = -\frac{1}{3}F$. Hence $X_i = -\frac{1}{3}(dF)_i$. It can be checked that such a X_i indeed satisfies $L(\boldsymbol{X})_{ij} = 0$. □

A.5 Generalisations of the De Turck trick

Here we give the proof of Proposition 17, from Section 6.1.1. We first recall some relevant background on the Ricci operator, following the presentation given in [53].

Recall that non-ellipticity of the linearised Ricci curvature tensor can be seen as a consequence of diffeomorphism invariance. Indeed, consider a one parameter family of diffeomorphisms, φ_t, then diffeomorphism-invariance of the Ricci operator is precisely the statement that

$$\mathrm{Ric}[\varphi_t^* \boldsymbol{h}]_{ij} = \varphi_t^* \mathrm{Ric}[\boldsymbol{h}]_{ij}.$$

Letting φ_t^* be generated by the vectorfield X^i, then the above linearises to give

$$D\mathrm{Ric}(\boldsymbol{h}) \cdot \mathcal{L}_{\boldsymbol{X}} h_{ij} = \mathcal{L}_{\boldsymbol{X}} \mathrm{Ric}[\boldsymbol{h}]_{ij},$$

or, equivalently

$$D\mathrm{Ric}(\boldsymbol{h}) \circ \delta_{\boldsymbol{h}}^*(\boldsymbol{X}) = -\tfrac{1}{2}\mathcal{L}_{\boldsymbol{X}} \mathrm{Ric}[\boldsymbol{h}],$$

where recall that

$$\delta_{\boldsymbol{h}}^*(\boldsymbol{X}) \equiv -\tfrac{1}{2}(D_i X_j + D_j X_i).$$

Note that $D\mathrm{Ric}(\boldsymbol{h})$ is a second-order differential operator, while $\delta_{\boldsymbol{h}}^*$ is a first-order operator. Hence the symbol map, $\sigma_\xi[\cdot]$, of the left-hand-side is an homogeneous polynomial of ξ_i of degree 3. The symbol map of the right-hand-side, however, is linear in ξ_i. Hence, we see that

$$0 = \sigma_\xi[D\mathrm{Ric}(\boldsymbol{h}) \circ \delta_{\boldsymbol{h}}^*] = \sigma_\xi[D\mathrm{Ric}(\boldsymbol{h})] \cdot \sigma_\xi[\delta_{\boldsymbol{h}}],$$

and hence that im $\sigma_\xi[\delta_{\boldsymbol{h}}^*] \subset$ ker $\sigma_\xi[D\mathrm{Ric}(\boldsymbol{h})]$ implying that the operator $D\mathrm{Ric}(\boldsymbol{h})$ is not elliptic. [3] Similar considerations for the scalar curvature allow one to derive the contracted Bianchi identity as a consequence of diffeomorphism invariance —see [96].

The above discussion my be subsumed in the statement that the following complex (see [53]):

$$0 \to \Lambda^1(\mathcal{S}) \xrightarrow{\delta^*} \mathscr{S}^2(\mathcal{S}) \xrightarrow{D\mathrm{Ric}(\boldsymbol{h})} \mathscr{S}^2(\mathcal{S}) \xrightarrow{B} \Lambda^1(\mathcal{S}) \to 0$$

is elliptic —that is to say, the corresponding complex of symbol maps

$$0 \to \Lambda^1(\mathcal{S}) \xrightarrow{\sigma[\delta^*]} \mathscr{S}^2(\mathcal{S}) \xrightarrow{\sigma[D\mathrm{Ric}(\boldsymbol{h})]} \mathscr{S}^2(\mathcal{S}) \xrightarrow{\sigma[B]} \Lambda^1(\mathcal{S}) \to 0, \tag{A.5.1}$$

[2] Of course, addition of a constant leaves $X_i \equiv (df)_i$ unchanged.
[3] In fact, ker $\sigma_\xi[D\mathrm{Ric}(\boldsymbol{h})] =$ im $\sigma_\xi[\delta_{\boldsymbol{h}}^*]$ —see [53].

is exact. Recall that $B(\cdot)$ denotes the Bianchi operator, given by

$$B(\boldsymbol{\gamma})_i \equiv C(\boldsymbol{\gamma})_{ij}{}^j = \delta(\boldsymbol{\gamma})_i - d(\mathrm{tr}_{\boldsymbol{h}}\boldsymbol{\gamma})_i.$$

By standard theory of elliptic complexes (see [61], for example), the operator

$$D\mathrm{Ric}^{\mathring{\delta}^*} \equiv D\mathrm{Ric} + \mathring{\delta}^* \circ \mathring{B}$$

is elliptic, and this is precisely the linearisation (at $\mathring{\boldsymbol{h}}$) of the de Turck-reduced operator

$$\widetilde{\mathrm{Ric}}[\boldsymbol{h}]_{ij} \equiv \mathrm{Ric}[\boldsymbol{h}]_{ij} + \delta^*_{\boldsymbol{h}} \circ Q(\boldsymbol{h})$$

used in previous chapters.

Proposition. Let $\mathring{\mathcal{K}} : \Lambda^1(\mathcal{S}) \to \mathscr{S}^2(\mathcal{S})$ be an overdetermined-elliptic operator. Then, the operator $D\mathrm{Ric}^{\mathring{\mathcal{K}}} : \mathscr{S}^2(\mathcal{S}) \to \mathscr{S}^2(\mathcal{S})$, defined by

$$D\mathrm{Ric}^{\mathring{\mathcal{K}}} \equiv D\mathrm{Ric} - \tfrac{1}{2}\mathring{\mathcal{K}} \circ \mathring{B},$$

is elliptic if and only if $\mathring{B} \circ \mathring{\mathcal{K}}$ is elliptic.

Proof. "$\Leftarrow$": To see this, note that it is sufficient to verify injectivity of the symbol map. Accordingly, suppose that

$$\sigma_\xi[D\mathrm{Ric}^{\mathring{\mathcal{K}}}](\boldsymbol{\gamma}) = 0. \tag{A.5.2}$$

Then,

$$\begin{aligned} 0 &= \sigma_\xi[\mathring{B}]\sigma[D\mathrm{Ric}^{\mathring{\mathcal{K}}}](\boldsymbol{\gamma}) \\ &= \sigma_\xi[\mathring{B} \circ D\mathrm{Ric}](\boldsymbol{\gamma}) - \tfrac{1}{2}\sigma_\xi[\mathring{B} \circ \mathring{\mathcal{K}} \circ \mathring{B}](\boldsymbol{\gamma}) \\ &= -\tfrac{1}{2}\sigma_\xi[\mathring{B} \circ \mathring{\mathcal{K}}]\sigma_\xi[\mathring{B}](\boldsymbol{\gamma}), \end{aligned} \tag{A.5.3}$$

where we have repeatedly used the product property of the symbol map and in the third line we have used the fact that $\sigma_\xi[\mathring{B}]\sigma_\xi[D\mathrm{Ric}] = 0$. By assumption $\mathring{B} \circ \mathring{\mathcal{K}}$ is elliptic, hence its symbol map is injective and it follows from (A.5.3) that

$$\sigma_\xi[\mathring{B}](\boldsymbol{\gamma})_i = 0.$$

Substituting into (A.5.2) yields

$$\sigma_\xi[D\mathrm{Ric}](\boldsymbol{\gamma})_{ij} = 0, \tag{A.5.4a}$$

$$\sigma_\xi[\mathring{B}](\boldsymbol{\gamma})_i = 0. \tag{A.5.4b}$$

By exactness of the complex (A.5.1) at the second entry, we have from (A.5.4a) that $\gamma_{ij} = \sigma_\xi[\mathring{\delta}^*](\boldsymbol{X})_{ij}$ for some $X_i \in \Lambda^1(\mathcal{S})$. Substituting into (A.5.4b),

$$\sigma_\xi[\mathring{B} \circ \mathring{\delta}^*](\boldsymbol{X})_i = 0. \tag{A.5.5}$$

Now, $\mathring{\delta}^*$, $\mathring{B}$ are overdetermined and underdetermined elliptic, respectively —i.e. exactness of the the symbol complex at the first and last entries. Moreover, it turns out that $\mathring{B} \circ \mathring{\delta}^*$ is in fact elliptic;

the principal symbol is given by

$$\begin{aligned}\sigma_\xi[\mathring{B}\circ\mathring{\delta}^*](\boldsymbol{X})_i &= \sigma_\xi[\mathring{B}]\sigma_\xi[\mathring{\delta}^*](\boldsymbol{X})_i\\ &= -\tfrac{1}{2}\xi^j(\xi_j X_i + \xi_i X_j - \xi^k X_k \mathring{h}_{ij})\\ &= -\tfrac{1}{2}|\boldsymbol{\xi}|^2 X_i.\end{aligned}$$

Hence, we see that $X_i = 0$, and so $\gamma_{ij} = \sigma_\xi[\mathring{\delta}^*](\boldsymbol{X})_{ij} = 0$. Hence, $D\mathrm{Ric}^{\mathring{\delta}^*}$ is elliptic.

"$\Rightarrow$": Fix $\xi_i \neq 0$ and consider

$$\sigma_\xi[\mathring{B}\circ\mathring{\mathcal{K}}](\boldsymbol{X})_i = 0.$$

We want to show that $X_i = 0$. Now $\sigma_\xi[\mathring{B}]$ maps onto $T_p^*\mathcal{S}$ —i.e. exactness of (A.5.1) at the last entry— and hence $X_i = \sigma_\xi[\mathring{B}](\boldsymbol{\gamma})_{ij}$ for some (non-unique) γ_{ij}. Then,

$$\begin{aligned}0 = \sigma_\xi[\mathring{B}\circ\mathring{\mathcal{K}}](\boldsymbol{X})_i &= \sigma_\xi[\mathring{B}\circ\mathring{\mathcal{K}}]\sigma_\xi[\mathring{B}](\boldsymbol{\gamma})_i\\ &= \sigma_\xi[\mathring{B}]\sigma_\xi[\mathring{\mathcal{K}}\circ\mathring{B}](\boldsymbol{\gamma})_i\\ &= -2\sigma_\xi[\mathring{B}]\sigma_\xi[D\mathrm{Ric}^{\mathring{\mathcal{K}}} - D\mathrm{Ric}](\boldsymbol{\gamma})_i\\ &= -2\sigma_\xi[\mathring{B}\circ D\mathrm{Ric}^{\mathring{\mathcal{K}}}](\boldsymbol{\gamma})_i.\end{aligned}$$

Applying $\sigma_\xi[\mathring{\mathcal{K}}]$,

$$\begin{aligned}0 = \sigma_\xi[\mathring{\mathcal{K}}\circ\mathring{B}\circ D\mathrm{Ric}^{\mathring{\mathcal{K}}}](\boldsymbol{\gamma})_{ij} &= \sigma_\xi[(\mathring{\mathcal{K}}\circ\mathring{B})\circ D\mathrm{Ric}^{\mathring{\mathcal{K}}}](\boldsymbol{\gamma})_{ij}\\ &= \sigma_\xi[D\mathrm{Ric}^{\mathring{\mathcal{K}}}\circ(\mathring{\mathcal{K}}\circ\mathring{B})](\boldsymbol{\gamma})_{ij}\\ &= \sigma_\xi[D\mathrm{Ric}^{\mathring{\mathcal{K}}}]\sigma_\xi[\mathring{\mathcal{K}}\circ\mathring{B}](\boldsymbol{\gamma})_{ij}.\end{aligned}$$

Since $D\mathrm{Ric}^{\mathring{\mathcal{K}}}$ is elliptic (by assumption) we have that

$$0 = \sigma_\xi[\mathring{\mathcal{K}}\circ\mathring{B}](\boldsymbol{\gamma})_{ij} = \sigma_\xi[\mathring{\mathcal{K}}]\sigma_\xi[\mathring{B}](\boldsymbol{\gamma})_{ij}.$$

Using the fact that $\mathring{\mathcal{K}}$ overdetermined elliptic, we find that $X_i \equiv \sigma_\xi[\mathring{B}](\boldsymbol{\gamma})_i = 0$. The above reasoning holds for any $\xi_i \neq 0$. Hence, $\mathring{B}\circ\mathring{\mathcal{K}}$ is elliptic, as claimed. □

Remark 65. Note that if $\mathring{\mathcal{K}}$ were not overdetermined elliptic, then of course $\mathring{B}\circ\mathring{\mathcal{K}}$ could not be elliptic. Given $\mathring{\mathcal{K}}$ overdetermined elliptic, $\mathring{B}\circ\mathring{\mathcal{K}}$ is elliptic if and only if $\mathrm{Im}\ \sigma[\mathring{\mathcal{K}}]\cap\ker\ \sigma[\mathring{B}] = \{\boldsymbol{0}\}$. Ellipticity of the De Turck-reduced Ricci operator is recovered as a special case by setting $\mathring{\mathcal{K}} = -2\mathring{\delta}^*$; recall that $\mathring{B}\circ\mathring{\delta}^*(\equiv \frac{1}{2}\mathring{\Delta}_Y)$ is elliptic, a fact which was used in the above proof.

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